Section 19, Gamma-Poisson Frequency Process[142]

The single most important specific example of mixing frequency distributions, is mixing Poisson Frequency Distributions via a Gamma Distribution. Each insured in a portfolio is assumed to have a Poisson distribution with mean λ. Across the portfolio, λ is assumed to be distributed via a Gamma Distribution. Due to the mathematical properties of the Gamma and Poisson there are some specific relationships. For example, as will be discussed, the mixed distribution is a Negative Binomial Distribution.

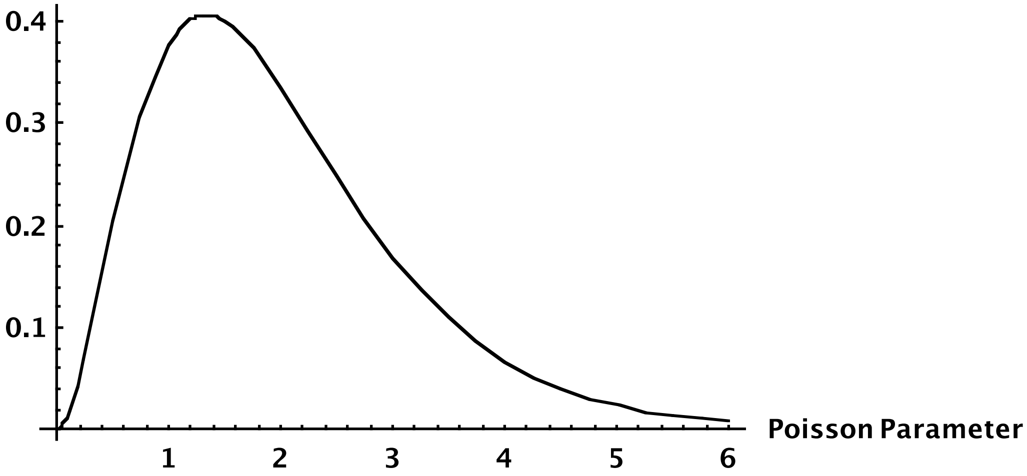
Prior Distribution:

The number of claims a particular policyholder makes in a year is assumed to be Poisson with mean λ. For example, the chance of having 6 claims is given by: $\lambda^6 e^{-\lambda} / 6!$

Assume the λ values of the portfolio of policyholders are Gamma distributed with $\alpha = 3$ and $\theta = 2/3$, and therefore probability density function:[143]

$$f(\lambda) = 1.6875\ \lambda^2\ e^{-1.5\lambda} \qquad \lambda \geq 0.$$

This prior Gamma Distribution of Poisson parameters is displayed below:



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[142] Section 6.3 of Loss Models.
Additional aspects of the Gamma-Poisson are discussed in "Mahler's Guide to Conjugate Priors."

[143] For the Gamma Distribution, $f(x) = \theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$.

The Prior Distribution Function is given in terms of the Incomplete Gamma Function: $F(\lambda) = \Gamma(3; 1.5\lambda)$. So for example, the a priori chance that the μ value lies between 4 and 5 is: $F(5) - F(4) = \Gamma(3; 7.5) - \Gamma(3; 6) = .9797 - .9380 = .0417$.

<u>Mixed Distribution</u>:

If we have a risk and do not know what type it is, in order to get the chance of having 6 claims, one would weight together the chances of having 6 claims, using the a priori probabilities and integrating from zero to infinity:[144]

$$\int_0^\infty \frac{\lambda^6\ e^{-\lambda}}{6!} f(\lambda)\ d\lambda = \int_0^\infty \frac{\lambda^6\ e^{-\lambda}}{6!} 1.6875\ \lambda^2\ e^{-1.5\lambda}\ d\lambda\ = 0.00234375\ \int_0^\infty \lambda^8\ e^{-2.5\lambda}\ d\lambda\,.$$

The integral can be written in terms of the (complete) Gamma function:

$$\int_0^\infty \lambda^{\alpha-1}\ e^{-\lambda/\theta}\ \ d\lambda\ = \Gamma(\alpha)\theta^\alpha.$$

Thus $\int_0^\infty \lambda^8\ e^{-2.5\lambda}\ d\lambda = \Gamma(9)\ 2.5^{-9} = (8!)\ (0.4)^9 \cong 10.57$.

Thus the chance of having 6 claims $\cong$ (0.00234375) (10.57) $\cong$ 2.5%.

More generally, if the distribution of Poisson parameters λ is given by a Gamma distribution $f(\lambda) = \theta^{-\alpha}\lambda^{\alpha-1}\ e^{-\lambda/\theta}/\ \Gamma(\alpha)$, and we compute the chance of having n accidents by integrating from zero to infinity:

$$\int_0^\infty \frac{\lambda^n\ e^{-\lambda}}{n!} f(\lambda)\ d\lambda = \int_0^\infty \frac{\lambda^6\ e^{-\lambda}}{6!}\ \frac{\lambda^{\alpha-1}\ e^{-\lambda/\theta}}{\theta^\alpha\ \Gamma(\alpha)}\ d\lambda = \frac{1}{n!\ \theta^\alpha\ \Gamma(\alpha)} \int_0^\infty \lambda^{n+\alpha-1}\ e^{-\lambda(1+1/\theta)}\ d\lambda\ =$$

$$\frac{1}{n!\ \theta^\alpha\ \Gamma(\alpha)}\ \frac{\Gamma(n+\alpha)}{(1\ +\ 1/\theta)^{n+\alpha}} = \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!}\ \frac{\theta^n}{(1\ +\ \theta)^{n+\alpha}}\,.$$

The mixed distribution is in the form of the Negative Binomial distribution with parameters $r = \alpha$ and $\beta = \theta$:

Probability of n accidents = $\frac{r(r+1)...(r+x-1)}{x!}\ \frac{\beta^x}{(1+\beta)^{x+r}}$.

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[144] Note the way both the Gamma and the Poisson have factors involving powers of λ and $e^{-\lambda}$ and these similar factors combine in the product.

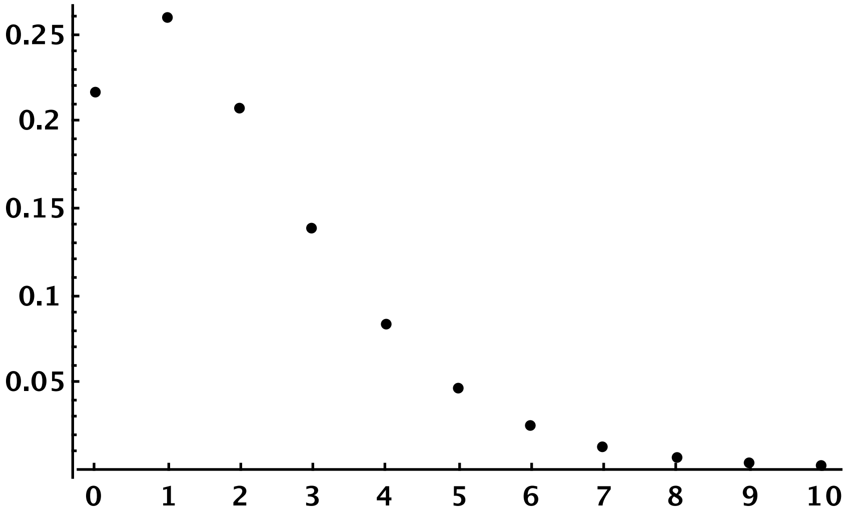
For the specific case dealt with previously: n = 6, α = 3 and θ = 2/3.

Therefore, the mixed Negative Binomial Distribution has parameters r = α = 3 and $\beta = \theta$ = 2/3.

Thus the chance of having 6 claims is: $\frac{(3)(4)(5)(6)(7)(8)}{6!} \frac{(2/3)^6}{(1 + 2/3)^{6+3}}$ = 2.477%.

This is the same result as calculated above.

This mixed Negative Binomial Distribution is displayed below, through 10 claims:



On the exam, one should <u>not</u> go through the calculation above. Rather remember that the mixed distribution is a Negative Binomial.

When Poissons are mixed via a Gamma Distribution,
the mixed distribution is always a Negative Binomial Distribution,
with r = α = shape parameter of the Gamma and
$\beta = \theta$ = scale parameter of the Gamma.

r goes with alpha, beta rhymes with theta.

Note that the overall (a priori) mean can be computed in either one of two ways.
First one can weight together the means for each type of risk, using the a priori probabilities. This is $E[\lambda]$ = the mean of the prior Gamma = $\alpha\theta = 3(2/3) = 2$. Alternately, one can compute the mean of the mixed distribution: the mean of a Negative Binomial is $r\beta = 3(2/3) = 2$. Of course the two results match.

Exponential-Poisson:[145]

It is important to note that **the Exponential distribution is a special case of the Gamma distribution, for $\alpha = 1$**.

For the important special case $\alpha = 1$, we have an Exponential distribution of λ: $f(\lambda) = e^{-\lambda/\theta}/\theta$, $\lambda \geq 0$.

The mixed distribution is a Negative Binomial Distribution with $r = 1$ and $\beta = \theta$.

For the Exponential-Poisson, the mixed distribution is a Geometric Distribution with $\beta = \theta$.

Mixed Distribution for the Gamma-Poisson, When Observing Several Years of Data:

One can observe for a period of time longer than a year. If an insured has a Poisson parameter of λ for each individual year, with λ the same for each year, and the years are independent, then for example one has a Poisson parameter of 7λ for 7 years. The chances of such an insured having a given number of claims over 7 years is given by a Poisson with parameter 7λ. For a portfolio of insureds, each of its Poisson parameters is multiplied by 7. This is mathematically just like inflation.

If before their each being multiplied by 7, the Poisson parameters follow a Gamma distribution with parameter α and θ, then after being multiplied by 7 they follow a Gamma with parameters α and 7θ.[146] Thus the mixed distribution for 7 years of data is given by a Negative Binomial with parameters $r = \alpha$ and $\beta = 7\theta$.

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[145] See for example 3/11/01, Q.27.

[146] Under uniform inflation, the scale parameter of the Gamma Distribution is multiplied by the inflation factor. See “Mahler’s Guide to Loss Distributions.”

In general, **if one observes a Gamma-Poisson situation for Y years, and each insured's Poisson parameter does not change over time, then the distribution of Poisson parameters for Y years is given by a Gamma Distribution with parameters α and $Y\theta$, and the mixed distribution for Y years of data is given by a Negative Binomial Distribution, with parameters $r = \alpha$ and $\beta = Y\theta$.**[147]

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. The distribution of λ over the portfolio of insureds is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
What is the mean annual claim frequency for the portfolio of insureds?
[Solution: The mean annual claims frequency = mean of the (prior) Gamma $= \alpha\theta = (3)(.01) = 3\%$.]

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. For each insured, λ does not change over time. For each insured, the numbers of claims in one year is independent of the number of claims in another year. The distribution of λ over the portfolio of insureds is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
An insured is picked at random and observed for 9 years.
What is the chance of observing exactly 4 claims from this insured?
[Solution: The mixed distribution for 9 years of data is given by a Negative Binomial Distribution with parameters $r = \alpha = 3$ and $\beta = Y\theta = (9)(.01) = .09$.

$$f(4) = \frac{(4+3-1)!}{4!\ 2!}\ \frac{0.09^4}{(1\ +\ 0.09)^{3+4}} = 0.054\%.]$$

If Lois has a low expected annual claim frequency, for example 2%, then over 9 years she has a Poisson Distribution with mean 18%. Her chance of having 4 claims during these nine years is: $0.18^4\ e^{-.18}/\ 24 = 0.004\%$.

If Hi has a very high expected annual claim frequency, for example 20%, then over 9 years he has a Poisson Distribution with mean 180%. His chance of having 4 claims during these nine years is: $1.8^4\ e^{-1.8}/\ 24 = 7.23\%$.

Drivers such as Lois with a low λ in one year are assumed to have the same low λ every year. Such good drivers have an extremely small chance of having four claims in 9 years.

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[147] "Each insured's Poisson parameter does not change over time." If Alan's lambda is 4% this year, it is 4% next year, and every year. Similarly, if Bonnie's lambda is 3% this year , then it is 3% every year.
Unless stated otherwise, on the exam assume lambda does <u>not</u> vary over time.

Drivers such as Hi with a very high λ in one year are assumed to have the same high λ every year. Such drivers have a significant chance of having four claims in 9 years. It is such very bad drivers which contribute significantly to the 0.054% probability of four claims in 9 years for an insured picked at random.

This situation in which for a given insured λ is the same over time, contrasts with that in which λ changes randomly each year.

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. For each insured, λ changes each year at random; the λ in one year is independent of the λ in another year.

The distribution of λ is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
An insured is picked at random and observed for 9 years.
What is the chance of observing exactly 4 claims from this insured?
[Solution: The mixed distribution for 1 year of data is given by a Negative Binomial Distribution with parameters $r = \alpha = 3$ and $\beta = \theta = .01 = .01$. Over 9 years, we get a sum of 9 independent Negative Binomials, with $r = (9)(3) = 27$ and $\beta = .01$.

$$f(4) = \frac{(4+27-1)!}{4!\ 26!}\ \frac{0.01^4}{(1\ +\ 0.01)^{27+4}} = 0.00020.]$$

This is different than the Gamma-Poisson process in which we assume that the lambda for an individual insured is the same each year. For the Gamma-Poisson the β parameter is multiplied by Y, while here the r parameter is multiplied by Y. This situation in which instead λ changes each year is mathematically the same as if we assume an insured each year has a Negative Binomial Distribution.

For example, assume an insured has a Negative Binomial with parameters r and β. Assume the numbers of claims in one year is independent of the number of claims in another year. Then over Y years, we add up Y independent identically distributed Negative Binomials; over Y years, the frequency distribution for this insured is Negative Binomial with parameters Yr and β.

Exercise: Assume that the number of claims in a year for an insured has a Negative Binomial Distribution with parameters $r = 3$ and $\beta = 0.01$. What is the mean annual claim frequency?
[Solution: $r\beta = (3)(.01) = 3\%$.]

Exercise: Assume that the number of claims in a year for an insured has a Negative Binomial Distribution with parameters r = 3 and β = .01. The numbers of claims in one year is independent of the number of claims in another year. What is the chance of observing exactly 4 claims over 9 years from this insured?
[Solution: Over 9 years, the frequency distribution for this insured is Negative Binomial with parameters r = (9)(3) = 27 and β= .01.

$$f(4) = \frac{(4+27-1)!}{4!\ 26!}\ \frac{0.01^4}{(1\ +\ 0.01)^{27+4}} = 0.00020.]$$

Even though both situations had a 3% mean annual claim frequency, the probability of observing 4 claims over 9 years was higher in the Gamma-Poisson situation with λ the same each year for a given insured, than when we assumed λ changed each year or equivalently an insured had the same Negative Binomial Distribution each year. In the Gamma-Poisson situation with λ the same each year for a given insured, we were more likely to see extreme results such as 4 claims in 9 years, since there is a small probability of picking at random an insured with a high expected annual claim frequency, such as Hi with λ = 20%.

Thinning a Negative Binomial Distribution:

Since the Gamma-Poisson is one source of the Negative Binomial Distribution, it can be used to aid our understanding of the Negative Binomial Distribution.

For example, assume we have a Negative Binomial Distribution with r = 4 and β = 2.
We can think of that as resulting from a mixture of Poisson Distributions, with λ distributed via a Gamma Distribution with α = 4 and θ = 2.[148]

Assume frequency and severity are independent, and that 30% of losses are "large."
Then for each insured, his large losses are Poisson with mean .3λ. If λ is distributed via a Gamma with α = 4 and θ = 2, then .3λ is distributed via a Gamma with α = 4 and θ = (0.3)(2) = 0.6.[149]
The large losses are a Gamma-Poisson Process, and therefore, across the whole portfolio, the distribution of large losses is Negative Binomial, with r = 4 and β = 0.6.

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[148] While this may not be real world situation that the Negative Binomial is modeling, since the results are mathematically identical, we can assume it is for the purpose of deriving general mathematical results.
[149] When a variable is Gamma Distributed, then a constant times that variable is also Gamma Distributed, with the same shape parameter, but with the scale parameter multiplied by that constant. See the discussion of uniform inflation in "Mahler's Guide to Loss Distributions."

In this manner one can show, as has been discussed previously, that if losses are Negative Binomial with parameters r and β, then if we take a fraction t of all the losses in a manner independent of frequency, then these selected losses are Negative Binomial with parameters r and $t\beta$.[150]

Returning to the example, the small losses for an individual insured are Poisson with mean $.7\lambda$. Since λ is Gamma distributed, $.7\lambda$ is distributed via a Gamma with $\alpha = 4$ and $\theta = (.7)(2) = 1.4$. Therefore, across the whole portfolio, the distribution of small losses is Negative Binomial, with $r = 4$ and $\beta = 1.4$.

Thus as in the Poisson situation, the overall process has been thinned into two similar processes. However, unlike the Poisson case, these two Negative Binomials are <u>not</u> independent.

If for example, we observe a lot of large losses, such as 5, it is more likely that the observation came from an insured with a large λ. This implies we are more likely to also have observed a higher than average number of small losses. The number of large losses and the number of small losses are positively correlated.[151]

<u>Correlation of Number of Small and Large Losses, Negative Binomial</u>:

Assume the number of losses follow a Negative Binomial Distribution with parameters r and β, and that "large" losses are t of all the losses. As previously, assume each insured is Poisson with mean λ, and λ is distributed via a Gamma with $\alpha = r$ and $\theta = \beta$.

Then the number of large losses is a Gamma-Poisson with $\alpha = r$ and $\theta = t\beta$.
Posterior to observing L large losses, the distribution of the mean frequency for large losses is Gamma with $\alpha = r + L$ and $1/\theta = 1/t\beta + 1 \Rightarrow \theta = t\beta/(1+ t\beta)$. Since the mean frequency of large losses is t times the mean frequency, posterior to observing L large losses, the distribution of the mean frequency is Gamma with $\alpha = r + L$ and $\theta = \beta/(1+ t\beta)$.

Therefore, given we have observed L large losses, the small losses are Gamma-Poisson with $\alpha = r + L$ and $\theta = (1\text{-}t)\beta/(1+ t\beta)$.

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[150] This can be derived via probability generating functions. See Example 8.8 in <u>Loss Models</u>.

[151] In the case of thinning a Binomial, the number of large and small loses would be negatively correlated.

One computes the correlation between the number of small losses, S, and the number of large losses, L, as follows:

$$E[LS] = E_L[E[LS \mid L]] = E_L[L\, E[S \mid L]] = E_L\left[\frac{L(r + L)(1-t)\beta}{1 + t\beta}\right] =$$

$$\frac{(1-t)\beta}{1 + t\beta}\{rE_L[L] + E_L[L^2]\} = \frac{(1-t)\beta}{1 + t\beta}\{rrt\beta + rt\beta(1+t\beta)+(rt\beta)^2\} = (1-t)t\beta^2 r(1+r).$$[152]

$$Cov[L, S] = E[LS] - E[L]E[S] = (1-t)t\beta^2 r(1+r) - rt\beta r(1-t)\beta = \beta^2 rt(1-t).$$

$$Corr[L, S] = \frac{\beta^2\ r\ t\ (1-t)}{\sqrt{r\ t\ \beta\ (1+t\beta)\ r\ (1-t)\ \beta\{1 + (1-t)\beta\}}} = \frac{1}{\sqrt{(1 + \frac{1}{t\beta})(1 + \frac{1}{(1-t)\beta})}} > 0.$$

For example, assume we have a Negative Binomial Distribution with r = 4 and β = 2.
Assume frequency and severity are independent, and that 30% of losses are "large."
Then the number of large losses are Negative Binomial with r = 4 and β = 0.6, and the number of small losses are Negative Binomial with r = 4 and β = 1.4.
The correlation of the number of large and small losses is:

$$\frac{1}{\sqrt{(1 + \frac{1}{t\beta})(1 + \frac{1}{(1-t)\beta})}} = \frac{1}{\sqrt{(1 + 1/0.6)(1 + 1/1.4)}} = 0.468.$$

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[152] Large losses are Negative Binomial with parameters r and tβ. Thus, $E_L[L^2] = Var[L] + E[L]^2 = rt\beta(1+t\beta)+(rt\beta)^2$.

Problems:

Use the following information to answer the next 2 questions:

The number of claims a particular insured makes in a year is Poisson with mean λ.

$\lambda$for a particular insured remains the same each year.

The values of the Poisson parameter λ (for annual claim frequency) for the insureds in a portfolio follow a Gamma distribution, with parameters $\alpha = 3$ and $\theta = 1/12$.

19.1 (2 points) What is the chance that an insured picked at random from the portfolio will have no claims over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 45% but less than 50%
E. at least 50%

19.2 (2 points) What is the chance that an insured picked at random from the portfolio will have one claim over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 45% but less than 50%
E. at least 50%

19.3 (2 points) The distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.6 and variance 0.9.
The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.
Calculate the variance of this gamma distribution.
(A) 0.20 (B) 0.25 (C) 0.30 (D) 0.35 (E) 0.40

19.4 (2 points) The number of claims a particular policyholder makes in a year has a Poisson distribution with mean μ. The μ-values for policyholders follow a gamma distribution with variance equal to 0.3. The resulting distribution of policyholders by number of claims is a Negative Binomial with parameters r and β such that the variance is equal to 0.7.

What is the value of $r(1+\beta)$?

A. less than 0.90
B. at least 0.90 but less than 0.95
C. at least 0.95 but less than 1.00
D. at least 1.00 but less than 1.05
E. at least 1.05

Use the following information for the next 3 questions:

Assume that the number of claims for an individual insured is given by a Poisson distribution with mean (annual) claim frequency λ and variance λ. Also assume that the parameter λ varies for the different insureds, with λ following a Gamma distribution:

$g(\lambda) = \theta^{-\alpha}\lambda^{\alpha-1}\ e^{-\lambda/\theta} / \Gamma(\alpha)$, for $0 < \lambda < \infty$, with mean $\alpha\theta$ and variance $\alpha\theta^2$.

19.5 (2 points) An insured is picked at random and observed for one year.
What is the chance of observing 2 claims?

A. $\alpha\theta^2 / (1+\theta)^{\alpha+2}$

B. $\alpha(\alpha+1)\theta^2 / (1+\theta)^{\alpha+2}$

C. $\alpha(\alpha+1)\theta^2 / \{2(1+\theta)^{\alpha+2}\}$

D. $\alpha^2(\alpha+1)\theta^2 / \{6(1+\theta)^{\alpha+2}\}$

E. $\alpha^2(\alpha+1)(\alpha+2)\theta^2 / \{6(1+\theta)^{\alpha+2}\}$

19.6 (2 points) What is the unconditional mean frequency?

A. $\alpha\theta$ B. $(\alpha-1)\theta$ C. $\alpha(\alpha-1)\theta^2$ D. $\alpha(\alpha-1)\theta^2$ E. $\alpha(\alpha-1)(\alpha+1)\theta^2/2$

19.7 (3 points) What is the unconditional variance?

A. $\alpha\theta^2$ B. $\alpha\theta + \alpha\theta^2$ C. $\alpha\theta + \alpha^2\theta^2$ D. $\alpha^2\theta^2$ E. $\alpha(\alpha+1)\theta$

Use the following information for the next 8 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 0.2 and variance 0.016.
The denominations of coins are randomly distributed: 50% of the coins are worth 5;
30% of the coins are worth 10; and 20% of the coins are worth 25.

19.8 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the tenth minute of today's walk.
(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

19.9 (3 points) Calculate the probability that Clumsy Klem loses exactly two coins during the first 10 minutes of today's walk.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

19.10 (4 points) Calculate the probability that the worth of the coins Clumsy Klem loses during his one-hour walk today is greater than 300.
A. 1% B. 3% C. 5% D. 7% E. 9%

19.11 (2 points) Calculate the probability that the sum of the worth of the coins Clumsy Klem loses during his one-hour walks each day for the next 5 days is greater than 900.
A. 1% B. 3% C. 5% D. 7% E. 9%

19.12 (2 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and possibly coins of other denominations?
A. 31% B. 33% C. 35% D. 37% E. 39%

19.13 (3 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and no coins of other denominations?
A. 11.6% B. 12.0% C. 12.4% D. 12.8% E. 13.2%

19.14 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. What is the probability that A + B = 3?
A. 0.2% B. 0.4% C. 0.6% D. 0.8% E. 1.0%

19.15 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. Let C be the number of coins Clumsy Klem loses during the first minute of his walk the day after tomorrow. What is the probability that A + B + C = 2?
A. 8% B. 10% C. 12% D. 14% E. 16%

19.16 (2 points) For an insurance portfolio the distribution of the number of claims a particular policyholder makes in a year is Poisson with mean λ.

The λ-values of the policyholders follow the Gamma distribution, with parameters $\alpha = 4$, and $\theta = 1/9$.

The probability that a policyholder chosen at random will experience x claims is given by which of the following?

A. $\frac{(x+3)!}{x!\ 3!}\ 0.9^4\ 0.1^x$

B. $\frac{(x+3)!}{x!\ 3!}\ 0.1^4\ 0.9^x$

C. $\frac{(x+8)!}{x!\ 8!}\ 0.75^4\ 0.25^x$

D. $\frac{(x+8)!}{x!\ 8!}\ 0.25^4\ 0.75^x$

E. None of A, B, C, or D.

19.17 (2 points) The number of claims a particular policyholder makes in a year has a Poisson distribution with mean λ. The λ-values for policyholders follow a Gamma distribution. This Gamma Distribution has a variance equal to one quarter that of the resulting Negative Binomial distribution of policyholders by number of claims.

What is the value of the β parameter of this Negative Binomial Distribution?

A. 1/6 B. 1/5 C. 1/4 D. 1/3 E. Can not be determined

19.18 (1 point) Use the following information:

- The random variable representing the number of claims for a single policyholder follows a Poisson distribution.
- For a portfolio of policyholders, the Poisson parameters follow a Gamma distribution representing the heterogeneity of risks within that portfolio.
- The random variable representing the number of claims in a year of a policyholder, chosen at random, follows a Negative Binomial distribution with parameters: $r = 4$ and $\beta = 3/17$.

Determine the variance of the Gamma distribution.

(A) .110 (B) .115 (C) .120 (D) .125 (E) .130

19.19 (2 points) Tom will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density $\lambda e^{-\lambda/1.4}/1.96$.

(ii) He then generates x from the Poisson distribution with mean λ.

(iii) He repeats the process 99,999 more times: first generating a value λ, then generating x from the Poisson distribution with mean λ.

Calculate the expected number of Tom's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

19.20 (2 points) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

19.21 (2 points) Dick will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density $\lambda\ e^{-\lambda/1.4}/1.96$.

(ii) He will then generate 100,000 independent values from the Poisson distribution with mean λ.

Calculate the expected number of Dick's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

19.22 (2 points) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

19.23 (1 point) Harry will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density

$\lambda\ e^{-\lambda/1.4}/1.96$.

(ii) He then generates x from the Poisson distribution with mean λ.

(iii) He will then copy 99,999 times this value of x.

Calculate the expected number of Harry's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

19.24 (1 point) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

Use the following information for the next 7 questions:

- The number of vehicles arriving at an amusement park per day is Poisson with mean λ.
- λ varies from day to day via a Gamma Distribution with $\alpha = 40$ and $\theta = 10$.
- The value of λ on one day is independent of the value of λ on another day.
- The number of people leaving each vehicle is:
  1 + a Negative Binomial Distribution with $r = 1.6$ and $\beta = 6$.
- The amount of money spent at the amusement park by each person is
  LogNormal with $\mu = 5$ and $\sigma = 0.8$.

19.25 (1 point) What is the variance of the number of vehicles that will show up tomorrow at the amusement park?

A. 4,000 B. 4,400 C. 4,800 D. 5,200 E. 5,600

19.26 (1 point) What is the variance of the number of vehicles that will show up over the next 7 days at the amusement park?

A. 25,000 B. 27,000 C. 29,000 D. 31,000 E. 33,000

19.27 (2 points) What is the variance of the number of people that will show up tomorrow at the amusement park?

A. 480,000 B. 490,000 C. 500,000 D. 510,000 E. 520,000

19.28 (1 point) What is the variance of the number of people that will show up over the next 7 days at the amusement park?

A. 2.8 million B. 3.0 million C. 3.2 million D. 3.4 million E. 3.6 million

19.29 (3 points) What is the standard deviation of the money spent tomorrow at the amusement park?

A. 150,000 B. 160,000 C. 170,000 D. 180,000 E. 190,000

19.30 (1 point) What is the standard deviation of the money spent over the next 7 days at the amusement park?

A. 360,000 B. 370,000 C. 380,000 D. 390,000 E. 400,000

19.31 (2 points) You simulate the amount of the money spent over the next 7 days at the amusement park. You run this simulation a total of 1000 times.
How many runs do you expect in which less than 5 million is spent?

A. 1 B. 2 C. 3 D. 4 E. 5

Use the following information to answer the next 3 questions:
The number of claims a particular policyholder makes in a year is Poisson. The values of the Poisson parameter (for annual claim frequency) for the individual policyholders in a portfolio of 10,000 follow a Gamma distribution, with parameters $\alpha = 4$ and $\theta = 0.1$.

You observe this portfolio for one year and divide it into three groups based on how many claims you observe for each policyholder: Group A: Those with no claims.
Group B: Those with one claim. Group C: Those with two or more claims.

19.32 (1 point) What is the expected size of Group A?
(A) 6200 (B) 6400 (C) 6600 (D) 6800 (E) 7000

19.33 (1 point) What is the expected size of Group B?
(A) 2400 (B) 2500 (C) 2600 (D) 2700 (E) 2800

19.34 (1 point) What is the expected size of Group C?
(A) 630 (B) 650 (C) 670 (D) 690 (E) 710

19.35 (4B, 11/96, Q.15) (2 points) You are given the following:

- The number of claims for a single policyholder follows a Poisson distribution with mean λ.
- λ follows a gamma distribution.
- The number of claims for a policyholder chosen at random follows a distribution with mean 0.10 and variance 0.15.

Determine the variance of the gamma distribution.
A. 0.05 B. 0.10 C. 0.15 D. 0.25 E. 0.30

19.36 (4B, 11/96, Q.26) (2 points) You are given the following:

- The probability that a single insured will produce 0 claims during the next exposure period is $e^{-\lambda}$.
- λ varies by insured and follows a distribution with density function
  $f(\lambda) = 36\lambda e^{-6\lambda}$, $0 < \lambda < \infty$.

Determine the probability that a randomly selected insured will produce 0 claims during the next exposure period.
A. Less than 0.72
B. At least 0.72, but less than 0.77
C. At least 0.77, but less than 0.82
D. At least 0.82, but less than 0.87
E. At least 0.87

19.37 (Course 3 Sample Exam, Q.12) The annual number of accidents for an individual driver has a Poisson distribution with mean λ.

The Poisson means, λ, of a heterogeneous population of drivers have a gamma distribution with mean 0.1 and variance 0.01.

Calculate the probability that a driver selected at random from the population will have 2 or more accidents in one year.

A. 1/121 B. 1/110 C. 1/100 D. 1/90 E. 1/81

19.38 (3, 5/00, Q.4) (2.5 points) You are given:

(i) The claim count N has a Poisson distribution with mean Λ.

(ii) Λ has a gamma distribution with mean 1 and variance 2.

Calculate the probability that N = 1.

(A) 0.19 (B) 0.24 (C) 0.31 (D) 0.34 (E) 0.37

19.39 (3, 5/01, Q.3) (2.5 points) Glen is practicing his simulation skills.

He generates 1000 values of the random variable X as follows:

(i) He generates the observed value λ from the gamma distribution with $\alpha = 2$ and $\theta = 1$ (hence with mean 2 and variance 2).

(ii) He then generates x from the Poisson distribution with mean λ.

(iii) He repeats the process 999 more times: first generating a value λ, then generating x from the Poisson distribution with mean λ.

(iv) The repetitions are mutually independent.

Calculate the expected number of times that his simulated value of X is 3.

(A) 75 (B) 100 (C) 125 (D) 150 (E) 175

19.40 (3, 5/01, Q.15) (2.5 points) An actuary for an automobile insurance company determines that the distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.2 and variance 0.4.

The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.

Calculate the variance of this gamma distribution.

(A) 0.20 (B) 0.25 (C) 0.30 (D) 0.35 (E) 0.40

19.41 (3, 11/01, Q.27) (2.5 points) On his walk to work, Lucky Tom finds coins on the ground at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 2 and variance 4.
Calculate the probability that Lucky Tom finds exactly one coin during the sixth minute of today's walk.

(A) 0.22 (B) 0.24 (C) 0.26 (D) 0.28 (E) 0.30

19.42 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first two minutes of today's walk.

(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

19.43 (3 points) In 3, 11/01, Q.27, let A = the number of coins that Lucky Tom finds during the first minute of today's walk. Let B = the number of coins that Lucky Tom finds during the first minute of tomorrow's walk. Calculate Prob[A + B = 1].

(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

19.44 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of today's walk.

A. Less than 4.5%
B. At least 4.5%, but less than 5.0%
C. At least 5.0%, but less than 5.5%
D. At least 5.5%, but less than 6.0%
E. At least 6.0%

19.45 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk, exactly two coins during the second minute of today's walk, and exactly three coins during the third minute of today's walk.

A. Less than 0.2%
B. At least 0.2%, but less than 0.3%
C. At least 0.3%, but less than 0.4%
D. At least 0.4%, but less than 0.5%
E. At least 0.5%

19.46 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk and exactly one coin during the fifth minute of tomorrow's walk.

(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

19.47 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first three minutes of today's walk and exactly one coin during the first three minutes of tomorrow's walk.

(A) 0.005 (B) 0.010 (C) 0.015 (D) 0.020 (E) 0.025

19.48 (3, 11/02, Q.5) (2.5 points) Actuaries have modeled auto windshield claim frequencies. They have concluded that the number of windshield claims filed per year per driver follows the Poisson distribution with parameter λ, where λ follows the gamma distribution with mean 3 and variance 3.
Calculate the probability that a driver selected at random will file no more than 1 windshield claim next year.
(A) 0.15 (B) 0.19 (C) 0.20 (D) 0.24 (E) 0.31

19.49 (CAS3, 11/03, Q.15) (2.5 points)
Two actuaries are simulating the number of automobile claims for a book of business.
For the population that they are studying:
i) The claim frequency for each individual driver has a Poisson distribution.
ii) The means of the Poisson distributions are distributed as a random variable, Λ.
iii) Λ has a gamma distribution.
In the first actuary's simulation, a driver is selected and one year's experience is generated. This process of selecting a driver and simulating one year is repeated N times.
In the second actuary's simulation, a driver is selected and N years of experience are generated for that driver.
Which of the following is/are true?

I. The ratio of the number of claims the first actuary simulates to the number of claims the second actuary simulates should tend towards 1 as N tends to infinity.

II. The ratio of the number of claims the first actuary simulates to the number of claims the second actuary simulates will equal 1, provided that the same uniform random numbers are used.

III. When the variances of the two sequences of claim counts are compared the first actuary's sequence will have a smaller variance because more random numbers are used in computing it.

A. I only B. I and II only C. I and III only D. II and III only E. None of I, II, or III is true

19.50 (CAS3, 5/05, Q.10) (2.5 points) Low Risk Insurance Company provides liability coverage to a population of 1,000 private passenger automobile drivers.
The number of claims during a given year from this population is Poisson distributed.
If a driver is selected at random from this population, his expected number of claims per year is a random variable with a Gamma distribution such that $\alpha = 2$ and $\theta = 1$.
Calculate the probability that a driver selected at random will not have a claim during the year.
A. 11.1% B. 13.5% C. 25.0% D. 33.3% E. 50.0%

19.51 (2 points) In CAS3, 5/05, Q.10, what is the probability that at most 265 of these 1000 drivers will not have a claim during the year?
A. 75% B. 78% C. 81% D. 84% E. 87%

19.52 (2 points) In CAS3, 5/05, Q.10, what is the probability that these 1000 drivers will have a total of more than 2020 claims during the year?
A. 31% B. 33% C. 35% D. 37% E. 39%

***19.53** (4 points) In CAS3, 5/05, Q.10, let A be the number of these 1000 drivers that have one claim during the year and B be the number of these 1000 drivers that have two claims during the year. Determine the correlation of A and B.*
A. -0.32 B. -0.30 C. -0.28 D. -0.26 E. -0.24

Section 20, Tails of Frequency Distributions

Actuaries are sometimes interested in the behavior of a frequency distribution as the number of claims gets very large.[153] The question of interest is how quickly the density and survival function go to zero as x approaches infinity. If the density and survival function go to zero more slowly, one describes that as a "heavier-tailed distribution."

Those frequency distributions which are heavier-tailed than the Geometric distribution are often considered to have heavy tails, while those lighter-tailed than Geometric are consider to have light tails.[154] There are number of general methods by which one can distinguish which distribution or empirical data set has the heavier tail. Lighter-tailed distributions have more moments that exist. For the frequency distributions on the exam all of the moments exist.

Nevertheless, the three common frequency distributions differ in their tail behavior. Since the Binomial has finite support, $f(x) = 0$ for $x > n$, it is very light-tailed. The Negative Binomial has its variance greater than its mean, so that the Negative Binomial is heavier-tailed than the Poisson which has its variance equal to its mean.

From lightest to heaviest tailed, the frequency distribution in the (a,b,0) class are:
Binomial, Poisson, Negative Binomial $r > 1$, Geometric, Negative Binomial $r < 1$.

Skewness:

The larger the skewness, the heavier-tailed the distribution. The Binomial distribution for $q > 0.5$ is skewed to the left (has negative skewness.) The Binomial distribution for $q < 0.5$, the Poisson distribution, and the Negative Binomial distribution are skewed to the right (have positive skewness); they have a few very large values and many smaller values. A symmetric distribution has zero skewness. Therefore, the Binomial Distribution for $q = 0.5$ has zero skewness.

Mean Residual Lives/ Mean Excess Loss:

As with loss distributions one can define the concept of the mean residual life.
The Mean Residual Life, e(x) is defined as:
e(x) = (average number of claims for those insureds with more than x claims) - x.
Thus we only count those insureds with more than x and only that part of each number of claims greater than x.[155] Heavier-tailed distributions have their mean residual life increase to infinity, while lighter-tailed distributions have their mean residual life approach a constant or decline to zero.

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[153] Actuaries are more commonly concerned with the tail behavior of loss distributions, as discussed in "Mahler's Guide to Loss Distributions."

[154] See Section 6.3 of Loss Models.

[155] Thus the Mean Residual Life is the mean of the frequency distribution truncated and shifted at x.

One complication is that for discrete distributions this definition is discontinuous at the integers. For example, assume we are interested in the mean residual life at 3. As we take the limit from below we include those insureds with 3 claims in our average; as we approach 3 from above, we don't include insureds with 3 claims in our average.

Define $e(3^-)$ as the limit as x approaches 3 from below of e(x). Similarly, one can define $e(3^+)$ as the limit as x approaches 3 from above of e(x). Then it turns out that $e(0^-)$ = mean, in analogy to the situation for continuous loss distributions. For purposes of comparing tail behavior of frequency distributions, one can use either $e(x^-)$ or $e(x^+)$. I will use the former, since the results using $e(x^-)$ are directly comparable to those for the continuous size of loss distributions. At integral values of x:

$$e(x^-) = \frac{\sum\limits_{i=x}^{\infty} (i-x)\, f(i)}{\sum\limits_{i=x}^{\infty} f(i)} = \frac{\sum\limits_{i=x}^{\infty} (i-x)\, f(i)}{S(x)}.$$

One can compute the mean residual life for the Geometric Distribution, letting $q = \beta/(1+\beta)$ and thus $1 - q = 1/(1+\beta)$:

$$e(x^-)\, S(x-1) = \sum_{i=x+1}^{\infty} (i-x)\, f(i) = \sum_{i=x+1}^{\infty} (i-x)\, \beta^i \,/\, (1+\beta)^{i+1} = \frac{1}{1+\beta} \sum_{i=x+1}^{\infty} (i-x)\, q^i =$$

$$\frac{1}{1+\beta} \left\{ \sum_{i=x+1}^{\infty} q^i + \sum_{i=x+2}^{\infty} q^i + \sum_{i=x+3}^{\infty} q^i + \ldots \right\} = (1-q) \left\{ \frac{q^{x+1}}{1-q} + \frac{q^{x+2}}{1-q} + \frac{q^{x+3}}{1-q} + \ldots \right\}$$

$$= q^{x+1} + q^{x+2} + q^{x+3} + \ldots = q^{x+1}/(1-q) = \{\beta/(1+\beta)\}^{x+1}(1+\beta) = \beta^{x+1}/(1+\beta)^x.$$

In a previous section, the survival function for the geometric distribution was computed as: $S(x) = \{\beta/(1+\beta)\}^{x+1}$. Therefore, $S(x-1) = \{\beta/(1+\beta)\}^x$.

Thus $e(x^-) = \dfrac{\beta^{x+1} \,/\, (1+\beta)^x}{\{\beta \,/\, (1+\beta)\}^x} = \beta.$

The mean residual life for the Geometric distribution is constant.[156] As discussed previously, the Geometric distribution is the discrete analog of the Exponential distribution which also has a constant mean residual life.[157]

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[156] $e(x^-) = \beta = E[X]$.

[157] The Exponential and Geometric distributions have constant mean residual lives due to their memoryless property as discussed in Section 6.3 of Loss Models.

As discussed previously, the Negative Binomial is the discrete analog of the Gamma Distribution. The tail behavior of the Negative Binomial is analogous to that of the Gamma.[158] The mean residual life for a Negative Binomial goes to a constant. For $r < 1$, $e(x^-)$ increases to β, the mean of the corresponding Geometric, while for $r > 1$, $e(x^-)$ decreases to β as x approaches infinity. For $r = 1$, one has the Geometric Distribution with $e(x^-)$ constant.

Using the relation between the Poisson Distribution and the Incomplete Gamma Function, it turns out that for the Poisson $e(x^-) = (\lambda - x) + \dfrac{\lambda^x e^{-\lambda}}{\Gamma(x)\,\Gamma(x;\,\lambda)}$.[159] The mean residual life $e(x^-)$ for the Poisson Distribution declines to zero as x approaches infinity.[160] [161] This is another way of seeing that the Poisson has a lighter tail than the Negative Binomial Distribution.

Summary:

Here are the common frequency distributions, arranged from lightest to heaviest righthand tail:

Frequency Distribution	Skewness	Righthand Tail Behavior	Tail Similar to
Binomial, $q > .5$	negative	Finite Support	
Binomial, $q = .5$	zero	Finite Support	
Binomial, $q < .5$	positive	Finite Support	
Poisson	positive	$e(x^-) \to 0$, approximately as $1/x$	Normal Distribution
Negative Binomial, $r > 1$	positive	$e(x^-)$ decreases to β	Gamma, $\alpha > 1$
Geometric	positive	$e(x^-)$ constant $= \beta$	Exponential
(Negative Binomial, $r = 1$)			(Gamma, $\alpha = 1$)
Negative Binomial, $r < 1$	positive	$e(x^-)$ increases to β	Gamma, $\alpha < 1$

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[158] See "Mahler's Guide to Loss Distributions", for a discussion of the mean residual life for the Gamma and other size of loss distributions. For a Gamma Distribution with $\alpha > 1$, e(x) decreases towards a horizontal asymptote θ. For a Gamma Distribution with $\alpha < 1$, e(x) increases towards a horizontal asymptote θ.

[159] For the Poisson $F(x) = 1 - \Gamma(x+1; \lambda)$.

[160] It turns out that $e(x^-) \cong \lambda / x$ for very large x. This is similar to the tail behavior for the Normal Distribution. While $e(x^-)$ declines to zero, $e(x^+)$ for the Poisson Distribution declines to one as x approaches infinity.

[161] This follows from the fact that the Poisson is a limit of Negative Binomial Distributions. For a sequence of Negative Binomial distributions with $r\beta = \lambda$ as $r \to \infty$ (and $\beta \to 0$), in the limit one approaches a Poisson Distribution with the mean λ. The tails of each Negative Binomial have $e(x^-)$ decreasing to β as x approaches infinity. As $\beta \to 0$, the limits of $e(x^-) \to 0$.

Skewness and Kurtosis of the Poisson versus the Negative Binomial:[162]

The Poisson has skewness: $\frac{1}{\sqrt{\lambda}}$.

The Negative Binomial has skewness: $\frac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$.

Therefore, if a Poisson and Negative Binomial have the same mean, $\lambda=r\beta$, then the ratio of the skewness of the Negative Binomial to that of the Poisson is: $\frac{1 + 2\beta}{\sqrt{1 + \beta}} > 1$.

The Poisson has kurtosis: $3 + 1/\lambda$.

The Negative Binomial has kurtosis: $3 + \frac{6\beta^2 + 6\beta + 1}{r\beta(1+\beta)}$.

Therefore, if a Poisson and Negative Binomial have the same mean, $\lambda=r\beta$, then the ratio of the kurtosis minus 3 of the Negative Binomial to that of the Poisson is:[163]

$$\frac{6\beta^2 + 6\beta + 1}{1 + \beta} > 1.$$

Tails of Compound Distributions:

Compound frequency distributions can have longer tails than either their primary or secondary distribution. If the primary distribution is the number of accidents, and the secondary distribution is the number of claims, then one can have a large number of claims either due to a large number of accidents, or an accident with a large number of claims, or a combination of the two. Thus there is more chance for an unusually large number of claims.

Generally the longer-tailed the primary distribution and the longer-tailed the secondary distribution, the longer-tailed the compound distribution. The skewness of a compound distribution can be rather large.

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[162] *See "The Negative Binomial and Poisson Distributions Compared," by Leroy J. Simon, PCAS 1960.*

[163] *The kurtosis minus 3 is sometimes called the excess.*

Tails of Mixed Distributions:

Mixed distributions can also have long tails. For example, the Gamma Mixture of Poissons is a Negative Binomial, with a longer tail than the Poisson. As with compound distributions, with mixed distributions there is more chance for an unusually large number of claims. One can either have a unusually large number of claims for a typical value of the parameter, have an unusual value of the parameter which corresponds to a large expected claim frequency, or a combination of the two. Generally the longer tailed the distribution type being mixed and the longer tailed the mixing distribution, the longer tailed the mixed distribution.

Tails of Aggregate Loss Distributions:

Actuaries commonly look at the combination of frequency and severity. This is termed the aggregate loss distribution. The tail behavior of this aggregate distribution is determined by the behavior of the heavier-tailed of the frequency and severity distributions.[164]

Since the common frequency distributions have tails that are similar to the Gamma Distribution or lighter and the common severity distributions for Casualty Insurance have tails at least as heavy as the Gamma, actuaries working on liability or workers compensation insurance are usually most concerned with the heaviness of the tail of the severity distribution. It is the rare extremely large claims that then are of concern.

However, natural catastrophes such as hurricanes or earthquakes can be examples where a large number of claims can be the concern.[165] *(Tens of thousands of homeowners claims, even limited to for example 1/4 million dollars each, can add up to a lot of money!) In that case the tail of the frequency distribution could be heavier than a Negative Binomial.*

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[164] See for example Panjer & Willmot, Insurance Risk Models.

[165] Natural catastrophes are now commonly modeled using simulation models that incorporate the science of the particular physical phenomenon and the particular distribution of insured exposures.

Problems:

20.1 (1 point) Which of the following frequency distributions have positive skewness?

1. Negative Binomial Distribution with $r = 3$, $\beta = 0.4$.
2. Poisson Distribution with $\lambda = 0.7$.
3. Binomial Distribution with $m = 3$, $q = 0.7$.

A. 1, 2 only
B. 1, 3 only
C. 2, 3 only
D. 1, 2, and 3
E. The correct answer is not given by A, B, C, or D.

Use the following information for the next five questions:

Five friends: Oleg Puller, Minnie Van, Bob Alou, Louis Liu, and Shelly Fish, are discussing studying for their next actuarial exam. They've counted 10,000 pages worth of readings and agree that on average they expect to find about 2000 "important ideas". However, they are debating how many of these pages there are expected to be with 3 or more important ideas.

20.2 (2 points) Oleg assumes the important ideas are distributed as a Binomial with $q = .04$ and $m = 5$.
How many pages should Oleg expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.3 (2 points) Minnie assumes the important ideas are distributed as a Poisson with $\lambda = .20$.
How many pages should Minnie expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.4 (2 points) Bob assumes the important ideas are distributed as a Negative Binomial with $\beta = 0.1$ and $r = 2$. How many pages should Bob expect to find with 3 or more important ideas?

A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.5 (3 points) Louis assumes the important ideas are distributed as a compound Poisson-Poisson distribution, with $\lambda_1 = 1$ and $\lambda_2 = 0.2$.

How many pages should Louis expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.6 (3 points) Shelly assumes the important ideas are distributed as a compound Poisson-Poisson distribution, with $\lambda_1 = 0.2$ and $\lambda_2 = 1$.

How many pages should Shelly expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.7 (3 points) Define Riemann's zeta function as: $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, $s > 1$.

Let the zeta distribution be: $f(x) = \frac{1}{x^{\rho+1}\,\zeta(\rho+1)}$, $x = 1, 2, 3, \ldots, \rho > 0$.

Determine the moments of the zeta distribution.

20.8 (4B, 5/99, Q.29) (2 points) A Bernoulli distribution, a Poisson distribution, and a uniform distribution each has mean 0.8. Rank their skewness from smallest to largest.

A. Bernoulli, uniform, Poisson
B. Poisson, Bernoulli, uniform
C. Poisson, uniform, Bernoulli
D. uniform, Bernoulli, Poisson
E. uniform, Poisson, Bernoulli

Section 21, Important Formulas and Ideas

Here are what I believe are the most important formulas and ideas from this study guide to know for the exam.

Basic Concepts (Section 2)

The mean is the average or expected value of the random variable.
The mode is the point at which the density function reaches its maximum.
The median, the 50th percentile, is the first value at which the distribution function is ≥ 0.5.
The 100pth percentile as the first value at which the distribution function ≥ p.
Variance = second central moment = $E[(X - E[X])^2] = E[X^2] - E[X]^2$.
Standard Deviation = Square Root of Variance.

Binomial Distribution (Section 3)

$$f(x) = f(x) = \binom{m}{x} q^x (1-q)^{m-x} = \frac{m!}{x!\,(m-x)!} q^x (1-q)^{m-x},\ 0 \le x \le m.$$

Mean = mq **Variance = mq(1-q)**

Probability Generating Function: $P(z) = \{1 + q(z-1)\}^m$
The Binomial Distribution for m =1 is a Bernoulli Distribution.
X is Binomial with parameters q and m_1, and Y is Binomial with parameters q and m_2,
X and Y independent, then X + Y is Binomial with parameters q and $m_1 + m_2$.

Poisson Distribution (Section 4)

$f(x) = \lambda^x e^{-\lambda} / x!$, $x \ge 0$

Mean = λ **Variance = λ**

Probability Generating Function: $P(z) = e^{\lambda(z-1)}$, $\lambda > 0$.

A Poisson is characterized by a constant independent claim intensity and vice versa.
The sum of two independent variables each of which is Poisson with parameters λ_1 and λ_2 is also Poisson, with parameter $\lambda_1 + \lambda_2$.

If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a Poisson.

Geometric Distribution (Section 5)

f(x) = $\frac{\beta^x}{(1+\beta)^{x+1}}$.

Mean = β **Variance = β(1+β)**

Probability Generating Function: $P(z) = \frac{1}{1 - \beta(z-1)}$.

For a Geometric Distribution, for n > 0, the chance of at least n claims is: $\left(\frac{\beta}{1+\beta}\right)^n$.

For a series of independent identical Bernoulli trials, the chance of the first success following x failures is given by a Geometric Distribution with mean
β=(chance of a failure) / (chance of a success).

Negative Binomial Distribution (Section 6)

f(x) = $\frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}$. **Mean = rβ** **Variance = rβ(1+β)**

Negative Binomial for r = 1 is a Geometric Distribution.

For the Negative Binomial Distribution with parameters β and r, with r integral, can be thought of as the sum of r independent Geometric distributions with parameter β.

If X is Negative Binomial with parameters β and r_1, and Y is Negative Binomial with parameters β and r_2, X and Y independent, then X + Y is Negative Binomial with parameters β and $r_1 + r_2$.

For a series of independent identical Bernoulli trials, the chance of success number r following x failures is given by a Negative Binomial Distribution with parameters r and
β=(chance of a failure) / (chance of a success).

Normal Approximation (Section 7)

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately:

$$\Phi\left[\frac{(j + 0.5) - \mu}{\sigma}\right] - \Phi\left[\frac{(i - 0.5) - \mu}{\sigma}\right].$$

Normal Distribution

$F(x) = \Phi((x-\mu)/\sigma)$

$$f(x) = \phi((x-\mu)/\sigma)/\sigma = \frac{\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]}{\sigma\sqrt{2\pi}}, -\infty < x < \infty. \qquad \phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}, -\infty < x < \infty.$$

Mean = μ **Variance = σ^2**

Skewness = 0 **(distribution is symmetric)** Kurtosis = 3

Skewness (Section 8)

Skewness = third central moment /STDDEV^3 = E[(X - E[X])^3]/STDDEV^3

$$= \{E[X^3] - 3\overline{X}E[X^2] + 2\overline{X}^3\} / \text{Variance}^{3/2}.$$

A symmetric distribution has zero skewness.

Binomial Distribution with q < 1/2 ⇔ positive skewness ⇔ skewed to the right.

Binomial Distribution q = 1/2 ⇔ symmetric ⇒ zero skewness.

Binomial Distribution q > 1/2 ⇔ negative skewness ⇔ skewed to the left.

Poisson and Negative Binomial have positive skewness.

Probability Generating Function (Section 9)

Probability Generating Function, p.g.f.:

$$P(z) = \text{Expected Value of } z^n = E[z^n] = \sum_{n=0}^{\infty} f(n)\, z^n.$$

The Probability Generating Function of the sum of independent frequencies is the product of the individual Probability Generating Functions.
The distribution determines the probability generating function and vice versa.
$f(n) = (d^n P(z) / dz^n)_{z=0} / n!$. $f(0) = P(0)$. $P'(1) = \text{Mean}$.

If a distribution is infinitely divisible, then if one takes the probability generating function to any positive power, one gets the probability generating function of another member of the same family of distributions. Examples of infinitely divisible distributions include: Poisson, Negative Binomial, Compound Poisson, Compound Negative Binomial, Normal, Gamma.

Factorial Moments (Section 10)

nth factorial moment $= \mu_{(n)} = E[X(X-1) .. (X+1-n)]$.

$\mu_{(n)} = (d^n P(z) / dz^n)_{z=1}$.

$P'(1) = E[X]$. $P''(1) = E[X(X-1)]$.

(a, b, 0) Class of Distributions (Section 11)

For each of these three frequency distributions: **$f(x+1) / f(x) = a + \{b / (x+1)\}$**, $x = 0, 1, ...$
where a and b depend on the parameters of the distribution:

Distribution	a	b	f(0)
Binomial	$-q/(1-q)$	$(m+1)q/(1-q)$	$(1-q)^m$
Poisson	0	λ	$e^{-\lambda}$
Negative Binomial	$\beta/(1+\beta)$	$(r-1)\beta/(1+\beta)$	$1/(1+\beta)^r$

Distribution	Mean	Variance	Variance Over Mean	
Binomial	mq	$mq(1-q)$	$1-q < 1$	**Variance < Mean**
Poisson	λ	λ	1	**Variance = Mean**
Negative Binomial	$r\beta$	$r\beta(1+\beta)$	$1+\beta > 1$	**Variance > Mean**

Distribution	Thinning by factor of t	Adding n independent, identical copies
Binomial	$q \to tq$	$m \to nm$
Poisson	$\lambda \to t\lambda$	$\lambda \to n\lambda$
Negative Binomial	$\beta \to t\beta$	$r \to nr$

For X and Y independent:

X	**Y**	**X+Y**
Binomial(q, m_1)	**Binomial(q, m_2)**	**Binomial(q, $m_1 + m_2$)**
Poisson(λ_1)	**Poisson(λ_2)**	**Poisson($\lambda_1 + \lambda_2$)**
Negative Binomial(β, r_1)	**Negative Bin.(β, r_2)**	**Negative Bin.(β, $r_1 + r_2$)**

Accident Profiles (Section 12)

For the Binomial, Poisson and Negative Binomial Distributions:
$(x+1) f(x+1) / f(x) = a(x + 1) + b$, where a and b depend on the parameters of the distribution.
$a < 0$ for the Binomial, $a = 0$ for the Poisson, and $a > 0$ for the Negative Binomial Distribution.

Thus if data is drawn from one of these three distributions, then we expect $(x+1) f(x+1) / f(x)$ for this data to be approximately linear with slope a; the sign of the slope, and thus the sign of a, distinguishes between these three distributions of the (a, b, 0) class.

Zero-Truncated Distributions (Section 13)

In general if f is a distribution on 0,1,2,3,..., then $h(x) = f(x) / \{1 - f(0)\}$ is a distribution on 1,2,3, ... We have the following three examples:

Distribution	Density of the Zero-Truncated Distribution	
Binomial	$\dfrac{\dfrac{m!\ q^x (1-q)^{m-x}}{x!\ (m-x)!}}{1 - (1-q)^m}$	$x = 1, 2, 3, \ldots, m$
Poisson	$\dfrac{e^{-\lambda}\ \lambda^x\ /\ x!}{1 - e^{-\lambda}}$	$x = 1, 2, 3, \ldots$
Negative Binomial	$\dfrac{\dfrac{r(r+1)\ldots(r+x-1)}{x!}\ \dfrac{\beta^x}{(1+\beta)^{x+r}}}{1 - 1/(1+\beta)^r}$	$x = 1, 2, 3, \ldots$

The moments of a zero-truncated distribution, h, are given in terms of those of the corresponding untruncated distribution, f, by: $E_h[X^n] = E_f[X^n] / \{1 - f(0)\}$.

The Logarithmic Distribution has support equal to the positive integers: $f(x) = \dfrac{\left(\dfrac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)}$.

The **(a,b,1) class of frequency distributions** is a generalization of the (a,b,0) class. As with the (a,b,0) class, the recursion formula: $f(x)/f(x-1) = a + b/x$ applies.
However, it need only apply now for $x \geq 2$, rather than $x \geq 1$.
Members of the (a,b,1) family include: all the members of the (a,b,0) family, the zero-truncated versions of those distributions: Zero-Truncated Binomial, Zero-Truncated Poisson, Extended Truncated Negative Binomial, and the Logarithmic Distribution.
In addition the (a,b,1) class includes the zero-modified distributions corresponding to these.

Zero-Modified Distributions (Section 14)

If f is a distribution on 0,1,2,3,..., and $0 < p_0^M < 1$,

then $h(0) = p_0^M$, $h(x) = f(x)\{1 - p_0^M\}/\{1 - f(0)\}$, x=1, 2 , 3..., is a distribution on 0,1,2,3, ...

The moments of a zero-modified distribution h are given in terms of those of f by

$E_h[X^n] = (1-p_0^M)\, E_f[X^n] / \{1 - f(0)\}$.

Compound Frequency Distributions (Section 15)

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.
p.g.f. of compound distribution = p.g.f. of primary dist.[p.g.f. of secondary dist.]

$P(z) = P_1[P_2(z)]$.

compound density at 0 = p.g.f. of the primary at the density at 0 of the secondary.

Moments of Compound Distributions (Section 16)

Mean of Compound Dist. = (Mean of Primary Dist.)(Mean of Sec. Dist.)
Variance of Compound Dist. = (Mean of Primary Dist.)(Var. of Sec. Dist.)
+ (Mean of Secondary Dist.)^2(Variance of Primary Dist.)

In the case of a Poisson primary distribution with mean λ, the variance of the compound distribution could be rewritten as: λ(2nd moment of Second. Dist.).

The third central moment of a compound Poisson distribution = λ(3rd moment of Sec. Dist.).

Mixed Frequency Distributions (Section 17)

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

The nth moment of a mixed distribution is the mixture of the nth moments.
First one mixes the moments, and then computes the variance of the mixture from its first and second moments.

The Probability Generating Function of the mixed distribution, is the mixture of the probability generating functions for specific values of the parameter.
For a mixture of Poissons, the variance is always greater than the mean.

Gamma Function (Section 18)

The (complete) **Gamma Function** is defined as:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt = \theta^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt \text{ , for } \alpha \geq 0, \theta \geq 0.$$

$\Gamma(\alpha) = (\alpha-1)!$ $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$$\int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt = \Gamma(\alpha)\,\theta^{\alpha}.$$

The **Incomplete Gamma Function** is defined as:

$$\Gamma(\alpha; x) = \int_0^x t^{\alpha-1} e^{-t}\, dt \;/\; \Gamma(\alpha).$$

Gamma-Poisson Frequency Process (Section 19)

If one mixes Poissons via a Gamma, then the mixed distribution is in the form of the Negative Binomial distribution with $r = \alpha$ and $\beta = \theta$.

If one mixes Poissons via a Gamma Distribution with parameters α and θ, then over a period of length Y, the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = Y\theta$.

For the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to:
mean of the Gamma + variance of the Gamma.

$\text{Var}[X] = E[\text{Var}[X \mid \lambda]] + \text{Var}[E[X \mid \lambda]]$. **Mixing increases the variance**.

Tails of Frequency Distributions (Section 20)

From lightest to heaviest tailed, the frequency distribution in the (a,b,0) class are:
Binomial, Poisson, Negative Binomial $r > 1$, Geometric, Negative Binomial $r < 1$.