NON-HOMOGENEOUS MARKOV PROCESSES

Again, we consider a stochastic process $\{M_n,\ n = 0,1,2,3,...\}$ that takes on a finite number of possible values (states). Like in a homogeneous case, the transition probabilities are history independent:

$$Q_n^{(i,j)} = \Pr\left[M_{n+1} = j \middle| M_n = i, M_{n-1} = i_{n-1}, \cdots M_0 = i_0\right] = \Pr\left[M_{n+1} = j \middle| M_n = i\right]$$

In non-homogeneous case however, the transition probabilities depend upon time n of transition.

The transition probability matrix Q_n is the $r \times r$ matrix of (i,j) entries $Q_n^{(i,j)}$, where r is the number of possible states.

We let ${}_kQ_n^{(i,j)} = \Pr\left[M_{n+k} = j \mid M_n = i\right]$.

Then ${}_kQ_n$ is the $r \times r$ matrix of (i,j) probabilities ${}_kQ_n^{(i,j)}$.

Thus we have: ${}_kQ_n = Q_n \times Q_{n+1} \times \cdots \times Q_{n+k-1}$.

Notation:

- ${}_{l+1}C^{(i,j)}$ denotes the cash flow at time l+1 if the subject is in State #i at time l and State #j at time l+1.
- ${}_lC^{(i)}$ denotes the cash flow at time l if the subject is in State #i at time l .
- ${}_{k+1}v_n$ denotes the discounting from time n+k+1 to the present time n.

Theorem (actuarial present value of cash flows at transitions).
Suppose that a subject is now in State #s at time n. Then the actuarial present value, as seen from the time n, of all the cash flows during transitions from State #i to State #j, is given by:

$$APV_{s@n}\left(C^{(i,j)}\right) = \sum_{k=0}^{\infty}\left[{}_kQ_n^{(s,i)} Q_{n+k}^{(i,j)}\right]\left[{}_{n+k+1}C^{(i,j)}\right]\left[{}_{k+1}v_n\right]$$

Theorem (actuarial present value of cash flows while in states).
Suppose that a subject is now in State #s at time n. Then the actuarial present value, as seen from the time n, of all cash flows while in State #i, is given by:

$$APV_{s@n}\left(C^{(i)}\right) = \sum_{k=0}^{\infty}\left[{}_kQ_n^{(s,i)}\right]\left[{}_{n+k}C^{(i)}\right]\left[{}_kv_n\right]$$

♠Example 8.6.

In *Continuing Care Retirement Communities* (CCRC) model, residents may move among four states:

State #1- Independent Living,
State #2- Temporarily in the Health Center,
State #3- Permanently in the Health Center,
State #4- Gone.

The transitions probabilities are from the Illustrative Matrices in Section 3.1 of the paper by Daniel and cash flows are from the Illustrative Cash Flows in Section 3.2 of the paper.

The subject is in Independent Living at time 5. Find the actuarial present value of the cash flows resulting from future transitions from Temporarily in the Health Center to Permanently in the Health Center, using 25% interest.

Solution.

We need to find $APV_{1@5}\left(C^{(2,3)}\right)$.

Notice that ${}_k v_n = v^k = \left(\frac{1}{1.25}\right)^k = (0.8)^k$.

$$APV_{1@5}\left(C^{(2,3)}\right) = \sum_{k=0}^{\infty}\left[{}_k Q_5^{(1,2)} Q_{5+k}^{(2,3)}\right]\left[{}_{5+k+1}C^{(2,3)}\right]\left[{}_{k+1}v_5\right] =$$

$$= Q_5^{(1,2)} Q_6^{(2,3)} {}_7C^{(2,3)} v^2 + {}_2Q_5^{(1,2)} Q_7^{(2,3)} {}_8C^{(2,3)} v^3.$$

Since ${}_2Q_5 = Q_5 Q_6$, we have:

$${}_2Q_5^{(1,2)} = (.3 \quad .2 \quad .3 \quad .2) \times (.2 \quad .1 \quad 0 \quad 0) = 0.08.$$

Thus, finally we can compute the actuarial present value:

$$APV_{1@5}\left(C^{(2,3)}\right) = (0.2)(0.4)(67)(0.8)^2 + (0.08)(0.3)(77)(0.8)^3 = 4.3766.$$

♠