Mahler's Guide to Stochastic Models

Information in bold or sections whose title is in bold are more important for passing the exam.
Larger bold type indicates it is extremely important.
Information presented in italics (and sections whose titles are in italics) should not be needed to directly answer exam questions and should be skipped on first reading. It is provided to aid the reader's overall understanding of the subject, and to be useful in practical applications.
Highly Recommended problems are double underlined. Recommended problems are underlined.[1]

Solutions are posted at http://www.neas-seminars.com.

Table of Contents is continued on the next page.

[1] Note that problems include both some written by me and some from past exams. The latter are copyright by the Society of Actuaries and the Casualty Actuarial Society and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the SOA and CAS bear no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams.

Course 3 Exam Questions by Section of this Study Aid[2]

							CAS 3	**SOA 3**
Section	Sample	5/00	11/00	5/01	11/01	11/02	11/03	11/03
1								
2					10, 11		32	26
3								
4	23	2	23, 29			9, 20	31	11
5							12, 13	
6							18	
7								
8		4		3, 15	27	5		
9		10		4, 36	19, 30	15	30	20
10								
11								
12				37				
13								
14						30		
15	26				29			24
16								
17								
18								
19								

Sections 17-19 on Nonhomogeneous Markov Chains cover material that was added to the syllabus in 2005.
The CAS/SOA did not release the 5/02 and 5/03 exams.
From 5/00 to 5/03, the Course 3 Exam was jointly administered by the CAS and SOA.
Starting in 11/03, the CAS and SOA gave separate exams.

[2] Excluding any questions that are no longer on the syllabus.

	CAS 3	**CAS 3**	**SOA 3**	**CAS 3**	**SOA M**	**CAS 3**	**SOA M**	**CAS 3**
Section	5/04	11/04	11/04	5/05	5/05	11/05	11/05	5/06
1								
2		18, 19	16	39				
3						28		
4	31	17		7, 11	5, 24, 25	29, 31	8	
5				17	39			
6		21						32
7								
8				10				
9	26				6	27	7, 40	
10	15, 27		26	14		26		33
11								
12				13				
13								
14	18, 23	15		36	11		23	29
15			14			39		24
16					12		6	
17				34		23	4, 5	
18								
19								

The SOA did not release its 5/04 and 5/06 exams.

	CAS 3	**SOA M**	**CAS 3**	**SOA MLC**	**CAS 3**
Section	11/06	11/06	5/07	5/07	11/07
1					
2	26				
3	27	8	1	5, 26	1, 2
4		9			
5					
6	23				
7					
8					
9			2	6	3
10	28			25	
11		10			
12					
13				8	
14		14		16	36
15	38			17	
16					40
17	21, 22		28		
18		15			
19			40	15	

The SOA did not release its 11/07 and subsequent exams.

	CAS 3L	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**
Section	5/08	11/08	5/09	11/09	5/10	11/10
1						
2		2			13	
3						10
4			8, 9	10, 11		
5						
6						
7						
8						
9	12	3	10		14	12
10	10, 11	1			12	11
11						
12						
13						
14	20	19		9	11	8, 9
15	25	25	7, 16		19	17
16				16		
17			6		10	
18						
19						

The SOA has posted a file with 282 Exam MLC Sample Questions.
Those on Stochastic Models are:
9. SOA 3, 11/03, Q.11
15. SOA 3, 11/03, Q.20
18. SOA 3, 11/03, Q.24
19. SOA 3, 11/03, Q.26
38. SOA 3, 11/04, Q.14
39. 3, 11/02 Q. 9
44. 3, 11/02 Q. 15
48. 3, 11/02 Q. 20
52. SOA 3, 11/04, Q.16
54. 3, 11/02 Q. 30
71. SOA 3, 11/04, Q.26
74.-75. 3, 11/01, Q.10-11
81. 3, 11/01, Q.19
87. 3, 11/01, Q.27
89. 3, 11/01, Q.29
90. 3, 11/01, Q.30
101. 3, 5/01, Q.4
124. 3, 5/01, Q.37
134. SOA M, 5/05, Q. 5
137. SOA M, 5/05, Q. 6
143. 3, 11/00, Q.23
149. 3, 11/00, Q.29
151. SOA M, 5/05, Q. 11
152. SOA M, 5/05, Q. 12
164. SOA M, 5/05, Q. 24
165. SOA M, 5/05, Q. 25
179. SOA M, 11/05, Q. 4
180. SOA M, 11/05, Q. 5
181. SOA M, 11/05, Q. 6
182. SOA M, 11/05, Q. 7
183. SOA M, 11/05, Q. 8
195. SOA M, 11/05, Q. 23
205. SOA M, 11/05, Q. 40
211. SOA M, 11/06, Q. 8
212. SOA M, 11/06, Q. 9
213. SOA M, 11/06, Q. 10
217. SOA M, 11/06, Q. 14
218. SOA M, 11/06, Q. 15
250. to 260. Additional Exam M Sample Q.1 to Q.11 on Markov Chains (See my sections 17-19.)

Section 1, Introduction

The concepts in "Poisson processes (and mixture distributions)" by James W. Daniel and "Multi-State Transition Models with Actuarial Applications" by James W. Daniel are demonstrated.

In this area, the syllabus is the same for CAS Exam 3L and SOA Exam MLC.
"Mahler's Guide to Stochastic Models" covers CAS 3L Learning Objectives: A5, B, and C3.

Assume we let X(t) = the surplus of the Sari Insurance Company at time t.
Then as time changes the surplus changes randomly.
This is an example of a **Stochastic Process**.
For each time t, X(t) is a random variable.

If we only look at the surplus at each year end, then this would be a **discrete-time** stochastic process. If instead we were able to examine the surplus at any point in time, this would be a **continuous-time** stochastic process. Generally, continuous-time processes can be approximated by discrete-time processes, by taking very small time intervals in a discrete time process. For example, the difference between being able to examine the surplus of the Sari Insurance Company at the end of each day or at any time is unlikely to be of any practical importance.

A **stochastic process {X(t), t $\in$ T} is a collection of random variables.**
T is the index set of the stochastic process.

So if one can only look at the surplus at each year end, we would have T = {1, 2, 3,...} in units of years. We would have corresponding random variables X(1), X(2), X(3) , ..., the observed amounts of surplus at years end. If instead we were able to examine the surplus at any point in time, we would have T = {t >0} and X(t), t >0.

In the Surplus example, $-\infty < X(t) < \infty$, since the surplus can (in theory) be any real number.[3] *The set of possible values for the random variables is called the state space of the stochastic process. For the surplus example, the state space is the set of real numbers.*[4]

Exercise: The price of stock at the close of business each day is P(t). What type of stochastic process is this? What is the state space? What is the index set?
[Solution: This is a discrete-time process. The state space is the positive real numbers.
The index set is the positive integers (in units of days.)]

[3] Small or negative amounts of Surplus would indicate an insurer in serious trouble or insolvent.
[4] Ignoring as usual, that the smallest unit of currency is $.01.

There are two types of stochastic processes covered on CAS Exam 3L and SOA MLC:
1. Poisson Processes, including the nonhomogeneous and compound cases.
2. Markov Chains, including the nonhomogeneous case.
Brownian Motion is covered on joint Exam 3F/MFE.[5]
Poisson Processes and Brownian Motion are continuous time models.[6]
Markov Chains are discrete time models.

Poisson Distribution:

Support: x = 0,1,2,3... Parameters: $\lambda > 0$

D. f. : $F(x) = 1 - \Gamma(x+1 ; \lambda)$ *Incomplete Gamma Function*

P. d. f. : $\mathbf{f(x) = \lambda^x e^{-\lambda} / x!}$

Mean = λ

Variance = λ

Mode = largest integer in λ *(if* λ *is an integer then* $f(\lambda) = f(\lambda-1)$ *and both* λ *and* λ*-1 are modes.)*

The sum of two independent variables each of which is Poisson with parameters λ_1 **and** λ_2 **is also Poisson, with parameter** $\lambda_1 + \lambda_2$**.**

Exponential Distribution:

$\mathbf{F(x) = 1 - e^{-x/\theta}}$ $\mathbf{f(x) = e^{-x/\theta} / \theta, \ x > 0.}$

Mean = θ **Variance =** θ^2 Second Moment = $2\theta^2$

When an Exponential Distribution is truncated and shifted from below, in other words when looks at the nonzero payments excess of a deductible, one gets the same Exponential Distribution, due to its memoryless property.

[5] Brownian Motion is discussed in Derivatives Markets by Robert McDonald,
and is covered in “Mahler’s Guide to Financial Economics.”
The material in Derivatives Markets by Robert McDonald is not on CAS Exam 3L or SOA Exam MLC.
[6] Other stochastic processes not on the syllabus include continuous time Markov Processes.

Binomial Distribution:

Support: $x = 0,1,2,3...,m$ Parameters: $1 > q > 0$, $m \geq 1$.

P. d. f. : $$f(x) = \frac{m!\, q^x (1-q)^{m-x}}{x!\, (m-x)!} = \binom{m}{x} q^x (1-q)^{m-x}.$$

Mean = mq **Variance = mq(1-q)**

Mode = largest integer in mq + q (if mq + q is an integer, then f(mq + q) = f(mq + q- 1) and both mq + q and mq + q - 1 are modes.)

A Binomial Distribution with m = 1 is a Bernoulli Distribution.

The sum of m independent Bernoulli Distributions with the same q is a Binomial with parameters m and q.

The sum of two independent Binomials with parameters (m_1, q) and (m_2, q) is also Binomial with parameters $m_1 + m_2$ and q.

Normal Distribution:

Support: $\infty > x > -\infty$ Parameters: $\infty > \mu > -\infty$ (location parameter)

$\sigma > 0$ (scale parameter)

D. f. : $$F(x) = \Phi((x-\mu)/\sigma)$$

P. d. f. : $$f(x) = \phi\left[\frac{x - \mu}{\sigma}\right]/\sigma = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}. \qquad \phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}.$$

Mean = μ **Variance =** σ^2

Skewness =0 (distribution is symmetric)

Mode = μ Median = μ

Attached to the exam is a table of the Standard Normal Distribution, with $\mu = 0$ and $\sigma = 1$.

Normal Distribution Table

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z < z).
The value of z to the first decimal place is given in the left column.
The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Table continued on the next page

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z < z).
The value of z to the first decimal place is given in the left column.
The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Values of z for selected values of Pr(Z < z)						
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z < z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

On Exams 3F/MFE and MLC, the following rule applies to the use of the Normal Table:[7]

When using the normal distribution, choose the nearest z-value to find the probability, or if the probability is given, chose the nearest z-value. No interpolation should be used.

Example: If the given z-value is 0.759, and you need to find Pr(Z < 0.759) from the normal distribution table, then choose the probability value for z-value = 0.76; Pr(Z < 0.76) = 0.7764.

When using the Normal Approximation to a discrete distribution, use the continuity correction.

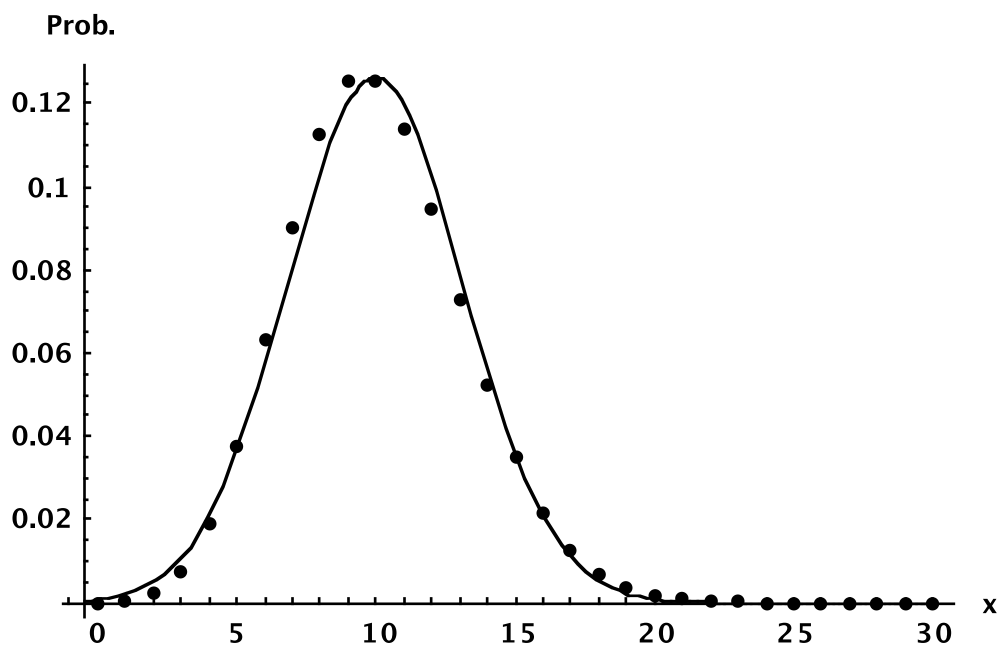
[7] While this rule does not apply to CAS Exam 3L, it would do no harm to follow it there as well.

The bottom of the Normal Table has selected percentiles of the Standard Normal Distribution.
For example, $\Phi[1.645] = 0.95$; 1.645 is the 95^{th} percentile of the Standard Normal Distribution.
$\Phi[1.960] = 0.975$; 1.960 is the 97.5^{th} percentile of the Standard Normal Distribution.

Normal Approximation:

The Poisson Distribution, with λ integral, is the sum of λ independent Poisson variables each with mean of one. Thus by the Central Limit Theorem, a Poisson Distribution can be approximated by a Normal Distribution with the same mean and variance.[8]

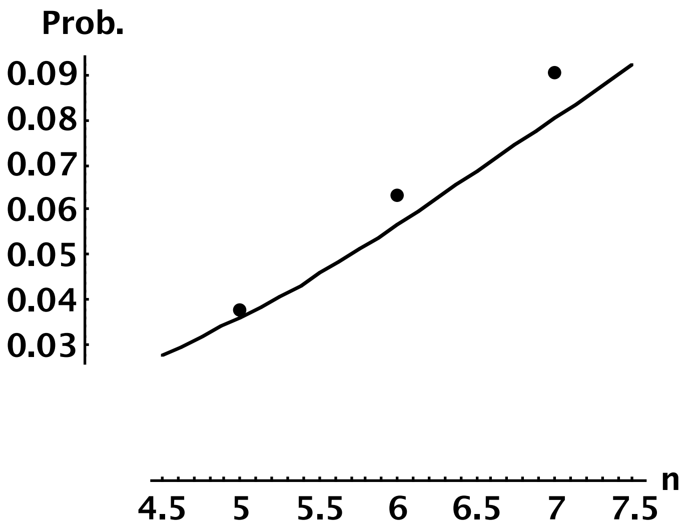
For example, here is the graph of a Poisson Distribution with $\lambda = 10$, and the approximating Normal Distribution with $\mu = 10$ and $\sigma = \sqrt{10} = 3.162$:



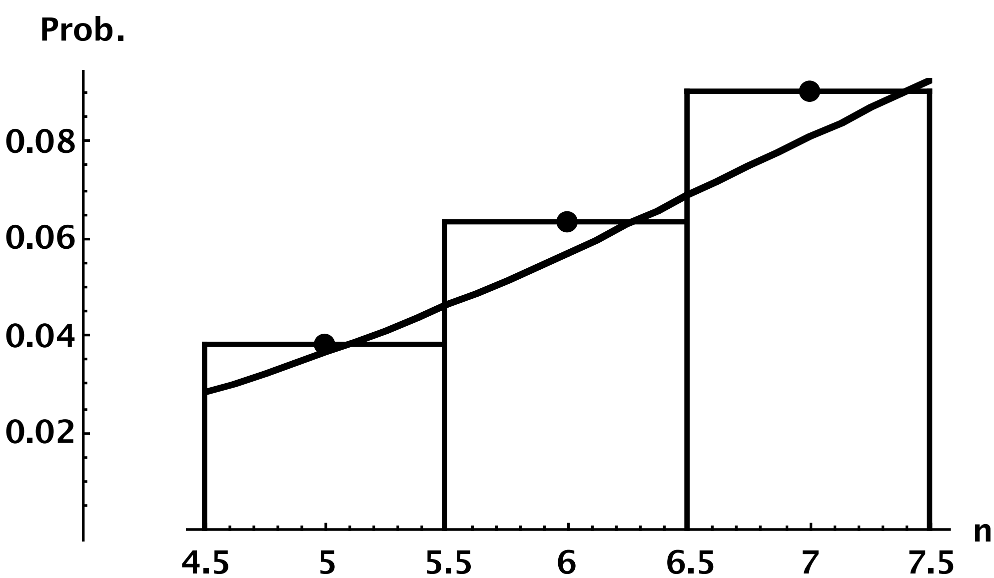
A typical use of the Normal Approximation would be to find the probability of observing a certain range of claims, where adding up the individual densities would be a lot of work. For example, given a certain distribution, what is the probability of at least 10 and no more than 20 claims. However, as an example, let us see how we can approximate the chance of 5, 6, or 7 claims.

[8] One can also approximate via a Normal Distribution the Binomial Distribution, Negative Binomial Distribution, and Compound Poisson Distributions.

Here is a graph of the densities of the Poisson Distribution with λ =10, at 5, 6, and 7, and the approximating Normal Distribution:



I have added rectangles to the above graph:



The first rectangle has height f(5) and width one, and thus area f(5).
The sum of the three rectangles is the exact answer: f(5) + f(6) + f(7).

In order to approximate the area of these three rectangles, I must go from 4.5 to 7.5 on the approximating continuous Normal Distribution: $F[7.5] - F[4.5] = \Phi[(7.5 - 10)/\sqrt{10}] - \Phi[(4.5 - 10)/\sqrt{10}]$
$= \Phi[-0.79] - \Phi[-1.74] = 0.2148 - 0.0409 = 0.1739$.

The exact answer is: $10^5 e^{-10}/5! + 10^6 e^{-10}/6! + 10^7 e^{-10}/7! = 0.1910$. In order to use the Normal Approximation, one must translate to the so called "Standard" Normal Distribution.[9] We therefore need to standardize the variables by subtracting the mean and dividing by the standard deviation.

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately: $\Phi[\frac{(j + 0.5) - \mu}{\sigma}] - \Phi[\frac{(i - 0.5) - \mu}{\sigma}]$.

One should **use the continuity correction whenever one is using the Normal Distribution in order to approximate the probability associated with a discrete distribution**.
Do not use the continuity correction when one is using the Normal Distribution in order to approximate continuous distributions, such as aggregate distributions or the Gamma Distribution.

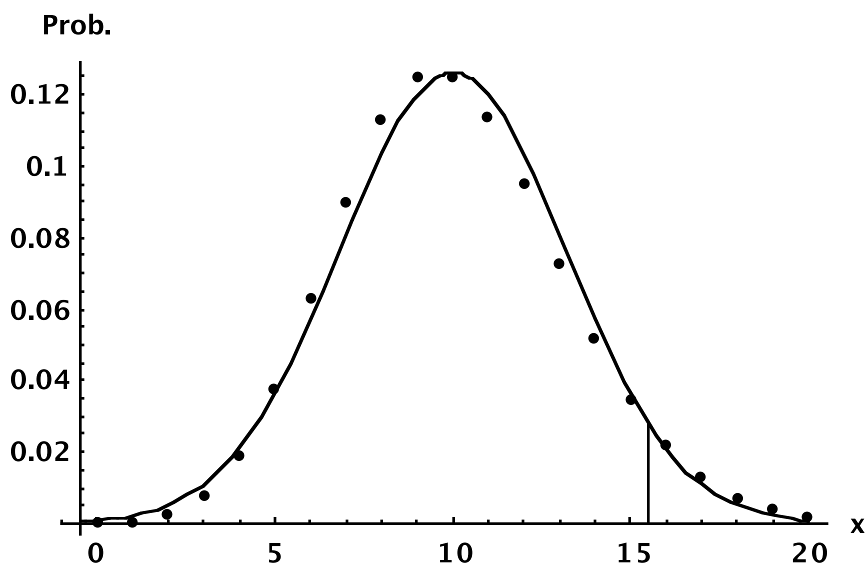
Exercise: Use the Normal Approximation in order to estimate the probability of observing at least 16 claims (more than 15 claims) from a Poisson Distribution with $\lambda = 10$.

[Solution: Mean = variance = 10. Prob[# claims ≥ 16] = 1 - Prob[# claims ≤ 15] ≅

$1 - \Phi[(15.5 - 10)/\sqrt{10}] = 1 - \Phi[1.74] = 1 - .9591 = 4.09\%$.

Comment: The exact answer is 4.87%.]

The area under the Normal Distribution and to the right of the vertical line at 15.5 is the approximation used in this exercise:



In order to get the probability of at least 16 (more than 15) on the discrete Poisson Distribution, one has to cover an interval starting at 15.5 on the real line for the continuous Normal Distribution.

[9] With mean zero and standard deviation one, in the table attached to the exam.

Diagrams for the Use of the Normal Approximation:

Some of you will find the following simple diagrams useful when applying the Normal Approximation to discrete distributions.

More than 15 claims ⇔ At least 16 claims ⇔ 16 claims or more

15 15.5 16

|⟶

Prob[More than 15 claims] $\cong$ 1 - Φ[(15.5 - μ)/σ].

Exercise: For a frequency distribution with mean 14 and standard deviation 2, using the Normal Approximation, what is the probability of at least 16 claims?

[Solution: Prob[At least 16 claims] = Prob[More than 15 claims] $\cong$ 1 - $\Phi[(15.5 - \mu)/\sigma] =$

1 - Φ[(15.5 - 14)/2] = 1 - Φ[0.75] = 1 - .7734 = 22.66%.]

Less than 12 claims ⇔ At most 11 claims ⇔ 11 claims or less

11 11.5 12

⟵|

Prob[Less than 12 claims] $\cong$ $\Phi[(11.5 - \mu)/\sigma]$.

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of at most 11 claims?

[Solution: Prob[At most 11 claims] = Prob[Less than 12 claims] $\cong$ $\Phi[(11.5 - \mu)/\sigma] =$

Φ[(11.5 - 10)/4] = Φ[.375] = 64.6%.]

At least 10 claims and at most 13 claims ⇔ More than 9 claims and less than 14 claims

9 9.5 10 11 12 13 13.5 14

◀———————————————▶

Prob[At least 10 claims and at most 13 claims] $\cong$ $\Phi[(13.5 - \mu)/\sigma] - \Phi[(9.5 - \mu)/\sigma]$.

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of more than 9 claims and less than 14 claims?
[Solution: Prob[more than 9 claims and less than 14 claims] =

Prob[At least 10 claims and at most 13 claims] $\cong \Phi[(13.5 - \mu)/\sigma] - \Phi[(9.5 - \mu)/\sigma] =$

$\Phi[(13.5 - 10)/4] - \Phi[(9.5 - 10)/4] = \Phi[.875] - \Phi[-.125] = .809 - .450 = 35.9\%$.]

Section 2, Poisson Processes

A (homogeneous) **Poisson Process has a constant claims rate (intensity) λ, and what happens on disjoint intervals (touching at an endpoint is okay) is independent**.
The number of claims observed in time interval (0, T) is given by the Poisson Distribution with mean $T\lambda$.[10] [11]

For example, assume we have a Poisson Process on (0, 5) with $\lambda = 0.03$.
Then the total number of claims is given by a Poisson Distribution with mean: (5)(0.03) = 0.15.

Exercise: For a Poisson Process with $\lambda = 0.7$, what is the chance of exactly 3 claims by time 2?
[Solution: Poisson Distribution with mean: (2)(0.7) = 1.4. $f(3) = 1.4^3 e^{-1.4}/3! = 11.3\%$.]

Thus if you understand the Poisson frequency distribution, you understand the (homogeneous) Poisson Process. However, in the case of a (homogeneous) Poisson Process one normally keeps track of both the total number of claims and the times they each occurred. If one had three claims, one would also want to know the three times at which they occurred.

Counting Processes:

A stochastic process N(t) on $t \geq 0$ is a counting process if represents the total number of events by time t. N(t) must satisfy:
1. $N(t) \geq 0$.
2. N(t) is integer valued.
3. If $s < t$, then $N(s) \leq N(t)$.
4. For $s < t$, N(t) - N(s) equals the number of events that have occurred in the interval (s, t].

Thus the state space of a counting process is the nonnegative integers.
A counting process is nondecreasing with time.

A **counting process is what an actuary would call a claims frequency process.** A Poisson Process is an example of a counting process. N(t) is the number of claims that have occurred by time t. N(t) - N(s) is the number of claims that have occurred in the interval (s, t], is the increment to the process.

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[10] Daniel refers to "events." I will usually refer to "claims", which is the most common application of these ideas for actuaries.
[11] *Note that by changing the time scale and therefore the claims intensity, one can always reduce to a mathematically equivalent situation where the interval is (0,1) .*

A claims frequency process has independent increments if the number of claims in two disjoint periods of time are independent of each other.

For example, for practical purposes the number of dental claims a large insurer gets this week are probably independent of the number of dental claims it gets next week. Thus we could model this frequency process with independent increments. However, the number of claims for the flu this week might be correlated with the number of claims for the flu last week.

A claims frequency has stationary increments if the distribution of the increment is a function of the width of the interval t - s, rather than s or t individually.

Stationary Increments $\Leftrightarrow$ the distribution of $N(s + \Delta) - N(s)$ is independent of s.

A (homogeneous) Poisson Process is a counting process with stationary and independent increments.

The assumption of stationary and independent increments is basically equivalent to asserting that at any point in time the process probabilistically restarts itself. In other words, the process has no memory.

If exposure levels are changing, then we would be unlikely to have stationary increments and therefore unlikely to have a (homogeneous) Poisson Process. Rather one would have a Nonhomogeneous Poisson Process, to be discussed subsequently.

Definition:

For a homogeneous Poisson Process with claims rate (intensity) λ:[12]

1. It is a counting process that starts at time = 0 with zero events; N(0) = 0.

2. It has independent increments.[13]

3. The increment $N[t + h] - N[t]$ is Poisson with mean λh.[14]

[12] See Definition 1.2 in "Poisson Processes" by Daniel.

[13] $N[t_1 + h_1] - N[t_1]$ and $N[t_2 + h_2] - N[t_2]$ are independent if $(t_1, t_1 + h_1]$ does not overlap with $(t_2, t_2 + h_2]$. Touching at endpoints is okay.

[14] Increments are stationary, with claims rate (intensity) lambda.
As discussed subsequently, for a nonhomogeneous Poisson Process, lambda is a function of time, and the increment is Poisson with mean equal to the integral of $\lambda(t)$ over the time interval.

o(h)

o(h) is a mathematical property a function may have.
A function f(x) is said to be o(h), if the limit as h approaches zero of f(h)/h is zero.[15]

In other words as h approaches zero, f(h) goes to zero more quickly than h.
For example, $f(x) = x^{1.5}$ is o(h). Thus x is of lower order of magnitude than $x^{1.5}$.
On the other hand, $\sqrt{x}$ is not o(h).

For a Poisson Process, the chance of having more than 1 claim in an interval of length h, is o(h). In other words, as the length of an interval approaches zero, the chance of having more than one claim in that interval also approaches zero, faster than the length of that interval approaches zero.
Prob[more than one claim in an interval of length h]/h, goes to zero as h goes to zero.

For a Poisson Process, the chance of having 1 claim in an interval of length h, is
λh + o(h). In other words, as h goes to zero,
Prob[one claim in interval of length h]/h, goes to λ as h goes to zero.

A function f(x) is equivalent to x as x goes to zero, f(x) ~ x, if the limit as h approaches zero of f(h)/h is one. For example, sin(x) ~ x, as x approaches zero.

A function f(x) is said to be O(h), if there is a positive constant K, such that for all sufficiently small h, $|f(h)/h| \le K$. For example, $f(x) = x(1 + a/x)^x$ is O(h), since the limit as x goes to zero of $(1 + a/x)^x$ is e^a. If a function is either o(h) or equivalent to h as h goes zero, then it is also O(h).

Another Definition of a Poisson Process:[16]

A Poisson Process is a counting process, N(t), such that:
N(0) = 0.
Stationary and independents increments.
Prob[N(h) = 1] = λh + o(h).
Prob[N(h) $\ge$ 2] = o(h).

[15] See Definition 5.2 of an Introduction to Probability Models by Sheldon M. Ross, not on the syllabus.
[16] See Definition 5.3 of an Introduction to Probability Models by Sheldon M. Ross, not on the syllabus.

Problems:

2.1 (1 point) A Poisson Process has a claims intensity of 0.3.
What is the probability of exactly 2 claims from time 0 to time 10?
A. 22% B. 24% C. 26% D. 28% E. 30%

2.2 (1 point) A Poisson Process has a claims intensity of 0.3.
Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 10 to time 20?
A. 22% B. 24% C. 26% D. 28% E. 30%

2.3 (3 points) A Poisson Process has a claims intensity of 0.3.
Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 6 to time 16?
A. 19% B. 21% C. 23% D. 25% E. 27%

2.4 (1 point) A Poisson Process has $\lambda = 0.6$.
What is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

2.5 (3 points) A Poisson Process has $\lambda = 0.6$.
If there are three claims from time 2 to time 8, what is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

2.6 (2 points) A Poisson Process has $\lambda = 0.6$.
What is the probability of 2 claims from time 0 to time 4 and 5 claims from time 0 to time 10?
A. 4.0% B. 4.5% C. 5.0% D. 5.5% E. 6.0%

2.7 (3 points) A Poisson Process has $\lambda = 0.6$. What is the probability of at least 1 claim from time 0 to time 4 and at least 3 claims from time 0 to time 6?
A. 69% B. 71% C. 73% D. 75% E. 77%

2.8 (1 point) A Poisson Process has $\lambda = 0.6$.
We observe 3 claims from time 0 to 1.
What is the probability of 4 claims from time 0 to 2?
A. 33% B. 35% C. 37% D. 39% E. 41%

2.9 (2 points) Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 15 is at least 60 but no more than 80?
Use the Normal Approximation.
(A) 73% (B) 75% (C) 77% (D) 79% (E) 81%

2.10 (3 points) Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 11 is greater than the number of claims from time 11 to 15?
Use the Normal Approximation.
(A) 87% (B) 89% (C) 91% (D) 93% (E) 95%

Use the following information for the next two questions:
The android Data is stranded on the planet Erehwon.
- Data uses energy uniformly at a rate of 10 gigajoules per year.
- If Data's stored energy reach 0, he ceases to function.
- Data gets his energy from dilithium crystals.
- Data gets 6 gigajoules of energy from each dilithium crystal.
- Data finds dilithium crystals at a Poisson rate of 2 per year.
- Data can store dilithium crystals without limit until needed.
- Data currently has 8 gigajoules of energy stored.

2.11 (4 points) What is the probability that Data ceases to function within the next 2.5 years?
(A) 30% (B) 33% (C) 36% (D) 39% (E) 42%

2.12 (3 points) What is the expected number of gigajoules of energy found by Data in the next 2.5 years?
(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

2.13 (3 points) Claims are given by a Poisson Process with $\lambda = 5$. What is probability that the number of claims between time 2 and 10 is greater than the number of claims from time 7 to 16?
Use the Normal Approximation.
(A) 17% (B) 19% (C) 21% (D) 23% (E) 25%

2.14 (1 point) Claims follow a homogeneous Poisson Process.
The average time between claims is 2.5 days.
How many whole number of days do we need to observe, in order to have at least a 95% probability of seeing at least one claim?
(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

2.15 (2 points) A Poisson Process has a claims intensity of 0.4 per day.
How many whole number of days do we need to observe, in order to have at least a 95% probability of seeing at least two claims?
(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

2.16 (1 point) A Poisson Process has a claims intensity of 0.04 per day. If there is at least 1 claim during a week, what is the probability that there are at least 2 claims during that week?
(A) 7% (B) 9% (C) 11% (D) 13% (E) 15%

2.17 (3 points) A Poisson Process has a claims intensity of 0.5.
What is the probability that the third claim occurs between time 5 and time 8?
(A) 10% (B) 15% (C) 20% (D) 25% (E) 30%

***2.18** (3 points) Data the Android gets new Emails at a Poisson rate of 15 per hour.*
Data checks for new Emails every x hours, where x has distribution $F(x) = 1 - 1/(27x^3)$, $x > 1/3$.
What is the variance of the number of new Emails Data finds when he checks?
(A) 18 (B)20 (C) 22 (D) 24 (E) 26

2.19 (3 points) Joe has four homework assignments: Math, English, History, and Chemistry.
He randomly chooses one of his assignments, and works on it until he completes it.
When Joe completes an assignment, he randomly chooses one of the remaining assignments, and works on it until completed.
Joe completes assignments at a Poisson rate of 1 assignment per 40 minutes.
Calculate the probability that Joe has completed his English assignment within 1 hour of starting his homework.
A. Less than 25%
B. At least 25%, but less than 30%
C. At least 30%, but less than 35%
D. At least 35%, but less than 40%
E. At least 40%

2.20 (3 points) For a certain company, losses follow a Poisson process with $\lambda = 3$ per year.
The amount of every loss is $100.
An insurance policy covers all losses in a year, subject to an annual aggregate deductible of $100.
(The insured pays the first $100 of loss during a year. The insurer pays any remaining losses.)
There have been no payments for this insurance policy during the first half of the year.
Calculate the expected claim payments for this insurance policy during the second half of the year.
(A) 90 (B) 100 (C) 110 (D) 120 (E) 130

2.21 (3 points) One has a Poisson Process with $\lambda = 25$.
Let A = the number of events that occur from time 1 to time 6.
Let B = the number of events that occur from time 3 to time 10.
Determine the correlation of A and B.
A. 30% B. 40% C. 50% D. 60% E. 70%

Use the following information for the next three questions:
One has a Poisson Process with $\lambda = 3$.
There is one event between time 0 to time 2.

2.22 (1 point) What is the expected number of events from time 0 to 4?
A. 5 B. 6 C. 7 D. 8 E. 9

2.23 (2 points) What is the variance of the number of events from time 0 to 4?
A. 5 B. 6 C. 7 D. 8 E. 9

2.24 (2 points) What is the probability that there are 5 events from time 0 to 4?
A. 5% B. 7% C. 9% D. 11% E. 13%

2.25 (2 points)
Policyholder calls to a call center follow a homogenous Poisson process with $\lambda = 250$ per day.
Using the normal approximation with continuity correction, calculate the probability of receiving at least 260 calls in a day.
A. Less than 27%
B. At least 27%, but less than 29%
C. At least 29%, but less than 31%
D. At least 31%, but less than 33%
E. At least 33%

2.26 (2 points) One has a Poisson Process with $\lambda = 10$ per hour.
You observe the process for 2 hours.
What is the probability that the total number of events during the first 20 minutes and the last 10 minutes combined is 6?
A. 10% B. 15% C. 20% D. 25% E. 30%

Use the following information for 3, 11/01 questions 10 and 11:
For a tyrannosaur with 10,000 calories stored:
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day.
If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.

2.27 (3, 11/01, Q.10) (2.5 points)
Calculate the probability that the tyrannosaur dies within the next 2.5 days.
(A) 0.30 (B) 0.40 (C) 0.50 (D) 0.60 (E) 0.70

2.28 (3, 11/01, Q.11) (2.5 points)
Calculate the expected calories eaten in the next 2.5 days.
(A) 17,800 (B) 18,800 (C) 19,800 (D) 20,800 (E) 21,800

2.29 (CAS3, 11/03, Q.32) (2.5 points) Daniel, in "Poisson Processes (and mixture distributions)", identifies four requirements that a counting process N(t) must satisfy.
Which of the following is <u>NOT</u> one of them?
A. N(t) must be greater than or equal to zero.
B. N(t) must be an integer.
C. If s<t, then N(s) must be less than or equal to N(t).
D. The number of events that occur in disjoint time intervals must be independent.
E. For s<t, N(t)-N(s) must equal the number of events that have occurred in the interval (s,t].

2.30 (SOA3, 11/03, Q.26) (2.5 points) A member of a high school math team is practicing for a contest. Her advisor has given her three practice problems: #1, #2, and #3.
She randomly chooses one of the problems, and works on it until she solves it. Then she randomly chooses one of the remaining unsolved problems, and works on it until solved.
Then she works on the last unsolved problem.
She solves problems at a Poisson rate of 1 problem per 5 minutes.
Calculate the probability that she has solved problem #3 within 10 minutes of starting the problems.
(A) 0.18 (B) 0.34 (C) 0.45 (D) 0.51 (E) 0.59

2.31 (CAS3, 11/04, Q.18) (2.5 points) Justin takes the train to work each day.
It takes 10 minutes for Justin to walk from home to the train station.
In order to get to work on time, Justin must board the train by 7:50 a.m.
Trains arrive at the station at a Poisson rate of 1 every 8 minutes.
What is the latest time he must leave home each morning so that he is on time for work at least 90% of the time?
A. 7:21 a.m. B. 7:22 a.m. C. 7:31 a.m. D. 7:32 a.m. E. 7:41 a.m.

2.32 (CAS3, 11/04, Q.19) (2.5 points) XYZ Insurance introduces a new policy and starts a sales contest for 1,000 of its agents. Each agent makes a sale of the new product at a Poisson rate of 1 per week. Once an agent has made 4 sales, he gets paid a bonus of $1,000. The contest ends after three weeks. Assuming 0% interest, what is the expected cost of the contest?
A. $18,988 B. $57,681 C. $168,031 D. $184,737 E. $352,768

2.33 (SOA3, 11/04, Q.16) (2.5 points) For a water reservoir:
(i) The present level is 4999 units.
(ii) 1000 units are used uniformly daily.
(iii) The only source of replenishment is rainfall.
(iv) The number of rainfalls follows a Poisson process with λ = 0.2 per day.
(v) The distribution of the amount of a rainfall is as follows:

Amount	Probability
8000	0.2
5000	0.8

(vi) The numbers and amounts of rainfalls are independent.
Calculate the probability that the reservoir will be empty sometime within the next 10 days.
(A) 0.27 (B) 0.37 (C) 0.39 (D) 0.48 (E) 0.50

2.34 (CAS3, 5/05, Q.39) (2.5 points) Longterm Insurance Company insures 100,000 drivers who have each been driving for at least five years.
Each driver gets "violations" at a Poisson rate of 0.5/year.
Currently, drivers with 1 or more violations in the past three years pay a premium of 1000.
Drivers with 0 violations in the past three years pay 850.
Your marketing department wants to change the pricing so that drivers with 2 or more violations in the past five years pay 1,000 and drivers with zero or one violations in the past five years pay X.
Find X so that the total premium revenue for your firm remains constant when this change is made.
A. Less than 900
B. At least 900, but less than 925
C. At least 925, but less than 950
D. At least 950, but less than 975
E. 975 or more
Note: I have slightly reworded this past exam question.

2.35 (CAS3, 11/06, Q.26) (2.5 points) Which of the following is/are true?

1. A counting process is said to possess independent increments if the number of events that occur between time s and t is independent of the number of events that occur between time s and t+u for all $u > 0$.
2. All Poisson processes have stationary and independent increments.
3. The assumption of stationary and independent increments is essentially equivalent to asserting that at any point in time the process probabilistically restarts itself.

A. 1 only B. 2 only C. 3 only D. 1 and 2 only E. 2 and 3 only

2.36 (CAS3L, 11/08, Q.2) (2.5 points) You are given the following:

- Hurricanes occur at a Poisson rate of 1/4 per week during the hurricane season.
- The hurricane season lasts for exactly 15 weeks.

Prior to the next hurricane season, a weather forecaster makes the statement,
"There will be at least three and no more than five hurricanes in the upcoming hurricane season."
Calculate the probability that this statement will be correct.
A. Less than 54%
B. At least 54%, but less than 56%
C. At least 56%, but less than 58%
D. At least 58%, but less than 60%
E. At least 60%

2.37 (CAS3L, 5/10, Q.13) (2.5 points)
You are given the following information regarding bank collapses:

- Bank collapses occur Monday through Thursday and are classified into two categories: severe and mild.
- Severe bank collapses occur only on Mondays and Tuesdays, and follow a Poisson distribution with $\lambda = 1$ on each of those days.
- Mild bank collapses can occur only on Wednesdays and Thursdays.
  If there was no more than one severe bank collapse earlier in the week, then mild bank collapses follow a Poisson distribution with $\lambda = 1$ for Wednesdays and Thursdays.
  If there was more than one severe bank collapse earlier in the week, then mild bank collapses follow a Poisson distribution with $\lambda = 2$ for Wednesdays and Thursdays.

Calculate the probability that no mild bank collapses occur during one week.
A. Less than 6.5%
B. At least 6.5% but less than 7.0%
C. At least 7.0% but less than 7.5%
D. At least 7.5% but less than 8.0%
E. At least 8.0%

Section 3, Interevent Times, Poisson Processes

Exercise: What is the probability that the waiting time until the first claim is less than or equal to 10 for a Poisson Process with $\lambda = 0.03$?
[Solution: The number of claims by time 10 is Poisson with mean: (10)(0.03) = 0.3.
Waiting time is $\leq 10 \Leftrightarrow$ At least one claim by t = 10. Prob[0 claims by time 10] = $e^{-0.3}$.
Prob[1 or more claims by time 10] = 1 - $e^{-0.3}$ = 25.9%.]

In this case, the distribution function of the waiting time until the first claim is: $F(t) = 1 - e^{-0.03t}$, an Exponential Distribution with mean 1/0.03. In general, for a Poisson Process with claims intensity λ, the **waiting time until the first claim has an Exponential Distribution with mean 1/λ.**
$F(t) = 1 - e^{-\lambda t}$.[17]

Interevent Times:

The interevent times (interarrival times) **are the times between events**.
V_1 is the waiting time until the first event. V_2 is the time from the first event until the second event.
V_j is the time from the j-1th event until the jth event.

Exercise: What is the probability that the waiting time from the third claim to the fourth claim, V_4, is less than or equal to 10 for a Poisson Process with $\lambda = 0.03$?

[Solution: Interevent time from 3rd to 4th claim $\leq 10 \Leftrightarrow$ # claims ≥ 1 in interval of length 10.
The number of claims over a time interval of length 10 is Poisson with mean: (10)(0.03) = 0.3.
Prob[0 claims] = $e^{-0.3}$. Prob[1 or more claims] = 1 - $e^{-0.3}$ = 25.9%.]

Due to the constant, independent claims intensity, the interevent time between the 1st and 2nd claim has the same distribution as the waiting time until the first claim. We can start a new Poisson Process, with claims intensity λ, when the 1st claim occurs; therefore, the wait until the next claim is Exponential with mean 1/λ. Similarly, we can start a new Poisson Process, with claims intensity λ, when the n^{th} claim occurs; therefore, the wait until the next claim is Exponential with mean 1/λ.
This interevent time, V_n, is independent of what happened before the n^{th} claim occurs.

For a Poisson Process with claims intensity λ, the interevent times are independent Exponential Distributions each with mean 1/λ.[18]

[17] The survival function is $\exp[-\lambda t]$, the same as that for a constant force of mortality λ. As long as we wait for the first claim, the mathematics is the same as the time until death for a constant force of mortality λ.
[18] Conversely, if the interevent times are independent, identically distributed Exponentials with mean θ, then the counting process is Poisson with hazard rate 1/θ.

For example, assume we have a Poisson Process with λ = .03, with an unrestricted time horizon. Then the waiting time to the first claim is Exponential, with θ = 1/.03 = 33.33.

The second interevent time is an independent Exponential, with θ = 33.33.
Each interevent time is an independent Exponential, with θ = 33.33.
The interevent times for a Poisson Distribution are those of the corresponding Poisson Process.
If claims follow a Poisson Distribution with mean annual frequency of .02, then each interevent time is an independent Exponential, with θ = 1/.02 = 50.

Event Times:

T_j is the time of the jth event (or claim). **$V_j = T_j - T_{j-1}$**. $T_n = V_1 + V_2 + ... + V_n$.

The Poisson Process is a random process. Sometimes you have to wait a long time for the next claim, and sometimes the next claim shows up right away. Similarly, T_3, how long you have to wait for the third claim to occur is random.

Fewer claims showing up. ⇔ Event time is larger.

More claims showing up. ⇔ Event time is smaller.

A key idea relates event times to the number of events by a given time.
For example, T_2, the time of the second event, is less than or equal to 10, if and only if there have been at least 2 events by time 10. $T_2 \le 10$. ⇔ $N(10) \ge 2$.

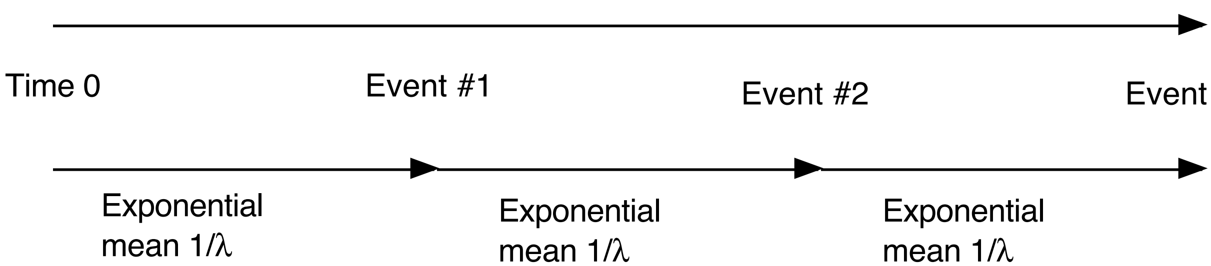
Exercise: Assume we have a Poisson Process with λ = 0.4.
What is the probability that $T_2 \le 10$?
[Solution: This is the same as the probability that we have observed at least 2 events by time 10. The number of events in the time period from 0 to 10 is Poisson Distributed with mean: (10)(.4) = 4. Therefore, the chance of zero events in this interval is e^{-4}. The chance of one event in this time interval is $4e^{-4}$. Thus the chance of at least 2 events in the time interval is: $1 - e^{-4} - 4e^{-4}$ = 90.8%.
Comment: Prob[$T_2 \le 10$] = Prob[$N(10) \ge 2$].]

In general, **$T_k \le t$. ⇔ $N(t) \ge k$.** **Prob[$T_k \le t$] = Prob[$N(t) \ge k$].**

Exercise: Assume we have a Poisson Process with $\lambda = .03$.
What is the probability that we have observed at least 4 claims by time 100?
[Solution: The number of claims in the time period from 0 to 100 is Poisson Distributed with mean (.03)(100) = 3. Therefore, the chance of zero claims in this interval is e^{-3}. The chance of one claim in this time interval is $3e^{-3}$. The chance of two claims in this time interval is $3^2e^{-3}/2$. The chance of three claims in this time interval is $3^3e^{-3}/6$. Thus the chance of at least 4 claims in the time interval is:
$1 - \{e^{-3} + 3e^{-3} + 3^2e^{-3}/2 + 3^3e^{-3}/6\} = 1 - .647232 = 0.352768$.
Comment: $\text{Prob}[T_4 \le 100] = \text{Prob}[N(100) \ge 4] = 0.352768$.]

Relationship to the Gamma Distribution:

The third event time has a Gamma Distribution: $\alpha = 3$, $\theta = 1/\lambda$.



The interevent times are independent, identically distributed Exponential Distributions with mean $1/\lambda$. The n^{th} event time, T_n, is the sum on n independent Exponential Variables, each with mean $1/\lambda$. Therefore, **the n^{th} event time, T_n, follows a Gamma Distribution with parameters $\alpha = n$ and $\theta = 1/\lambda$.**[19]

The mean time until the n^{th} claim is n/λ. Note that if we restrict the Poisson Process to (0,T), then there may be fewer than n claims; in other words the time until the n^{th} claim could be greater than T.

For example, assume we have a Poisson Process with $\lambda = .03$, with an unrestricted time horizon. Then the 4th event time is given by a Gamma Distribution with parameters $\alpha = 4$ and $\theta = 33.33$. The mean time of the 4th claim is the mean of this Gamma Distribution, $\alpha\theta = (4)(33.33) = 133.33$. T_4 follows a Gamma Distribution with $\alpha = 4$ and $\theta = 33.33$. Thus $F(100) = \Gamma[4; 100/33.33] = \Gamma[4; 3]$.

[19] The Gamma Distribution is discussed in a subsequent section.
See Definition and Properties 1.20 in Daniel.
The Gamma Distribution is parameterized as per Appendix A of Loss Models.

However, in the previous exercise, we have computed that Prob[$T_4 \le 100$] = 0.352768.

Thus we have shown that Γ[4; 3] = 0.352768, without the use of a computer!

A Formula for the Incomplete Gamma Function:

In general, assume claims are given by a Poisson Process with claims intensity λ. Then the claims in the interval from (0, t) are Poisson Distributed with mean tλ. One can calculate the chance that there are least n claims in two different ways. First, the chance of at least n claims is a sum of Poisson densities:

$$1 - F(n-1) = 1 - \sum_{i=0}^{n-1} e^{-\lambda t} (\lambda t)^i / i! = \sum_{i=n}^{\infty} e^{-\lambda t} (\lambda t)^i / i!.$$

On the other hand, the n^{th} event time is a Gamma Distribution with α = n and θ = 1/ λ.

Thus the n^{th} event time has distribution Γ[n ; λ t].

Comparing the two results when t = 1, the Incomplete Gamma Function with integer shape parameter λ can be written in terms of a sum of Poisson densities:[20]

$$\Gamma[n ; \lambda] = 1 - \sum_{i=0}^{n-1} e^{-\lambda} \lambda^i / i! = \sum_{i=n}^{\infty} e^{-\lambda} \lambda^i / i!.$$

For example, $\Gamma[3; 3.5] = 1 - \{e^{-3.5} + 3.5e^{-3.5} + 3.5^2e^{-3.5}/2\} = 1 - 0.320847 = 0.679153$.

We have also established a formula for the Distribution Function of a Poisson with mean λ, in terms of the Incomplete Gamma Function:

$$F(x) = \sum_{i=0}^{x} e^{-\lambda} \lambda^i / i! = 1 - \Gamma[x+1 ; \lambda].$$

[20] See Theorem A.1 in Appendix A of Loss Models, not on the syllabus.

Relationship to the Chi-Square Distribution:[21]

Since the Chi-Square Distribution is a special case of the Gamma Distribution, one can use a Chi-Square Table in order to estimate the Distribution Function of a Poisson.

A Chi-Square Distribution with ν degrees of freedom is a Gamma Distribution, as per Loss Models, with parameters $\alpha = \nu/2$ and $\theta = 2$.

Exercise: Use the following Chi-Square Table in order to estimate F(5) for a Poisson with mean 10.5.

Degrees	Significance Levels				
of Freedom	0.100	0.050	0.025	0.010	0.005
12	18.55	21.03	23.34	26.22	28.30

[Solution: $F(5) = 1 - \Gamma(5+1; 10.5) = 1 - \Gamma(6; 21/2) =$

1 - Chi-Square Distribution for 12 degrees of freedom at $21 \cong .05$.]

In general, for a Poisson with mean λ, F(x)

= 1 - Chi-Square Distribution with (2x+2) degrees of freedom at 2λ

= significance level of 2λ for a Chi-Square with (2x+2) degrees of freedom.

Time Until the Next Event:

Let T(x) = the time from x until the next event.

Since what happened before time x is independent of what happens after time x, we can start a new process at time x. Therefore, T(x) is Exponential with mean $1/\lambda$. $\Pr[T(x) > t] = e^{-\lambda t}$.

Exercise: Assume we have a Poisson Process with $\lambda = 0.03$.
What is the probability that the first event to occur after time 10 occurs by time 15?
[Solution: $\Pr[T(10) \le 5] = 1 - \exp[-(0.03)(5)] = 13.9\%$.]

[21] The Chi-Square Distribution is on the syllabus of CAS Exam 3L, but not SOA Exam MLC.

Problems:

3.1 (1 point) A Poisson Process has a claims intensity of 0.05.
What is the mean time until the first claim?
A. 5 B. 10 C. 15 D. 20 E. 25

3.2 (1 point) A Poisson Process has a claims intensity of 0.05.
What is the mean time until the tenth claim?
A. 50 B. 100 C. 200 D. 300 E. 400

3.3 (1 point) A Poisson Process has a claims intensity of 0.05.
What is the probability that the time until the first claim is greater than 35?
A. Less than 14%
B. At least 14%, but less than 15%
C. At least 15%, but less than 16%
D. At least 16%, but less than 17%
E. At least 17%

3.4 (1 point) A Poisson Process has a claims intensity of 0.05.
What is the probability that the time from the fifth claim to the sixth claim is less than 10?
A. Less than 36%
B. At least 36%, but less than 38%
C. At least 38%, but less than 40%
D. At least 40%, but less than 42%
E. At least 42%

3.5 (2 points) A Poisson Process has a claims intensity of 0.05.
What is the probability that the time from the eighth claim to the tenth claim is greater than 50?
A. Less than 28%
B. At least 28%, but less than 30%
C. At least 30%, but less than 32%
D. At least 32%, but less than 34%
E. At least 34%

3.6 (3 points) A Poisson Process has $\lambda = 0.9$.
What is the probability that the fourth event occurs between time 2 and time 5?
A. 40% B. 45% C. 50% D. 55% E. 60%

3.7 (2 points) For a claim number process you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function:
$F(t) = 1 - e^{-t/2}$, $t \geq 0$.
Determine the probability of exactly 3 claims in an interval of length 7.
A. Less than 23%
B. At least 23%, but less than 25%
C. At least 25%, but less than 27%
D. At least 27%, but less than 29%
E. At least 29%

3.8 (2 points) Claims follow a Poisson Process. The average time between claims is 5.
What is the probability that we have observed at least 2 claims by time 9?
A. Less than 53%
B. At least 53%, but less than 55%
C. At least 55%, but less than 57%
D. At least 57%, but less than 59%
E. At least 59%

3.9 (2 points) Claims follow a Poisson Process. The average time between claims is 5.
What is the probability that we have observed exactly 2 claims by time 9?
A. Less than 23%
B. At least 23%, but less than 25%
C. At least 25%, but less than 27%
D. At least 27%, but less than 29%
E. At least 29%

3.10 (2 points) Use the following information:
- Customers arrive at a subway token booth in accordance with a Poisson process with mean 4 per minute.
- There is one clerk and his service time is exponentially distributed with a mean of 10 seconds.

When Joe arrives at the token booth, no other customers are there.
What is the probability that Joe is done being served before another customer arrives?
(A) 40% (B) 45% (C) 50% (D) 55% (E) 60%

3.11 (2 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute.
What is the probability that during his one-hour walk tomorrow Tom finds his third coin during the second five minutes of his walk?
A. 40% B. 42% C. 44% D. 46% E. 48%

Use the following information for the next five questions:

- You and your friend George go together to the subway platform.
- George is waiting for the uptown train and you are waiting for the downtown train.
- Uptown trains arrive via a Poisson Process at a rate of 5 per hour.
- Downtown trains arrive via a Poisson Process at a rate of 5 per hour.
- The arrival of downtown and uptown trains are independent.

3.12 (2 points) What is the average time until the first one of you catches his train?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 10 minutes (E) 12 minutes

3.13 (2 points) What is the average time until the last one of you catches his train?
(A) 15 minutes (B) 18 minutes (C) 21 minutes (D) 24 minutes (E) 27 minutes

3.14 (2 points) What is the average time you wait on the platform without George?
(A) 6 minutes (B) 8 minutes (C) 10 minutes (D) 12 minutes (E) 14 minutes

3.15 (1 point) On average how many uptown trains pass while you wait on the platform?
(A) 1/2 (B) 3/4 (C) 1 (D) 3/2 (E) 2

3.16 (2 points) What is the probability that exactly three downtown trains pass while George waits on the platform?
(A) 2% (B) 3% (C) 4% (D) 5% (E) 6%

3.17 (1 point) Claims occur via a homogeneous Poisson Process. The expected waiting time until the first claim is 770 hours. If the claims intensity had been 5 times as large, what would have been the expected waiting time until the first claim?
A. 154 hours B. 765 hours C. 770 hours D. 3850 hours E. None of the above.

3.18 (2 points) Assume a driver's claim frequency is given by a Poisson distribution, with an average annual claim frequency of 8%.
What is the probability that it will be more than 25 years until this driver's first claim?
A. less than 10%
B. at least 10% but less than 11%
C. at least 11% but less than 12%
D. at least 12% but less than 13%
E. at least 13%

3.19 (2 points) For a Poisson Process, the average time between claims is 0.2.
What is the probability that we have observed at least 3 claims by time 1.2?
A. Less than 93%
B. At least 93%, but less than 95%
C. At least 95%, but less than 97%
D. At least 97%, but less than 99%
E. At least 99%

3.20 (3 points) Use the following information:
- In the late afternoon, customers arrive at a bagel shop via a Poisson Process at a rate of 20 per hour.
- Each customer first has his order filled and then pays for that order.
- The time to fill an order is exponentially distributed with a mean of 90 seconds.
- The time to pay for an order is exponentially distributed with a mean of 10 seconds.
- The times to fill an order and to pay for that order are independent.

When Mary arrives at the bagel shop, no other customers are there.
What is the probability that Mary is done getting her order and paying for it before another customer arrives?
(A) 63% (B) 65% (C) 67% (D) 69% (E) 71%

<u>**3.21**</u> (2 points) Buses arrive via a Poisson Process with λ = 6 per hour. Sandy arrives at 9:02.
Sandy is still waiting for a bus at 9:05.
What is the probability that the next bus arrives by 9:10?
A. 35% B. 40% C. 45% D. 50% E. 55%

3.22 (1 point) Debbie receives phone calls at work via a Poisson Process at a rate of 5 per hour.
Right after finishing a phone call, Debbie leaves her desk for 10 minutes in order to get a snack.
Determine the probability that Debbie received at least one call while away from her desk.
(A) 57% (B) 59% (C) 61% (D) 63% (E) 65%

***3.23** (2 points) One has a counting process N(t) with stationary and independent increments.*
$N(0) = 0$. $Prob[N(x) = 1] = 8x + o(x)$. $Prob[N(x) \geq 2] = o(x)$.
What is the variance of the time between the fourth and fifth events?
A. 1/64 B. 1/8 C. 1/4 D. 1/2 E. 1

Use the following information for the next five questions:
Bonnie and Clyde arrive at the bus terminal and each wait for their bus to leave.
Bonnie is going to Brighton and Clyde is going to Chelsea.
Brighton buses leave the terminal via a Poisson Process at a rate of 5 per hour.
Chelsea buses leave the terminal via a Poisson Process at a rate of 15 per hour.
Brighton buses and Chelsea buses are independent.

3.24 (2 points) On average how many Chelsea buses leave while Bonnie is at the terminal?
A. 1 B. 2 C. 3 D. 4 E. 5

3.25 (3 points) Determine the average time Bonnie waits if Clyde's bus leaves first.
A. 12 minutes B. 13 minutes C. 14 minutes D. 15 minutes E. 16 minutes

3.26 (2 points) What is the average time until the first one of them catches his or her bus?
A. 3 minutes B. 4 minutes C. 5 minutes D. 6 minutes E. 7 minutes

3.27 (2 points) What is the average time until the last one of them catches his or her bus?
A. 12 minutes B. 13 minutes C. 14 minutes D. 15 minutes E. 16 minutes

3.28 (2 points) What is the probability that exactly five Chelsea busses leave the terminal while Bonnie waits?
A. 5% B. 6% C. 7% D. 8% E. 9%

3.29 (2 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. What is the probability that the first two coins Lucky Tom finds during his one-hour walk today are found within one minute of each other?
A. 30% B. 35% C. 40% D. 45% E. 50%

3.30 (2 points) Trains arrive via a Poisson Process. The average time between trains is 10. Conditional on the first train arriving before time 15, what is the expected waiting time until the first train?
A. 5.7 B. 5.9 C. 6.1 D. 6.3 E. 6.5

3.31 *(3 points) A bus leaves at 10 minute intervals.*

Passengers arrive via a Poisson Process with $\lambda = 3$ per minute.

Let X be the total time in minutes spent waiting by all of the passengers who board a single bus. Determine the variance of X.

A. 250 B. 500 C. 750 D. 1000 E. 1250

Use the following information for the next two questions:

- Cautious Clarence wants to cross a road.
- Vehicles pass the spot where Clarence is waiting via a Poisson Process at a rate of one every 4 seconds.
- Clarence will wait until he can see that no vehicle will come by in the next 10 seconds.

3.32 (1 point) What is the probability that Clarence does not have to wait?
A. 6% B. 8% C. 10% D. 12% E. 14%

3.33 *(3 points) Calculate Clarence's average waiting time.*
A. 20 B. 25 C. 30 D. 35 E. 40

Use the following information for the next two questions:

- Mistakes in cell division occur via a Poisson Process with $\lambda = 2$ per year.
- An individual dies when 150 such mistakes have occurred.

3.34 (2 points) What is the variance of the lifetime of an individual?
A. Less than 30
B. At least 30, but less than 35
C. At least 35, but less than 40
D. At least 40, but less than 45
E. At least 45

3.35 (2 points) Using the Normal Approximation, estimate the probability that an individual survives to age 85.
A. 3% B. 5% C. 7% D. 9% E. 11%

3.36 *(3 points) Flip Wilson flips a penny once per minute.*
The chance of a head is p for each flip, independent of the other flips.
What is the distribution of the interevent times between heads?

Use the following information for the next three questions:
People visit an ATM via a Poisson process at a rate of 20 per hour.
There have been 10 people between 6:00 PM and 7:00 PM.
It is now 7:00 PM.

3.37 (1 point) At what time do you expect the 13^{th} person?
A. Before 7:05 PM
B. On or after 7:05 PM, but before 7:10 PM
C. On or after 7:10 PM, but before 7:15 PM
D. On or after 7:15 PM, but before 7:20 PM
E. On or after 7:20 PM

3.38 (1 point) What is the probability that the 11^{th} person arrives by 7:05 PM?
A. Less than 70%
B. At least 70%, but less than 75%
C. At least 75%, but less than 80%
D. At least 80%, but less than 85%
E. At least 85%

3.39 (2 points) What is the probability that the 13^{th} person arrives by 7:12 PM?
A. Less than 70%
B. At least 70%, but less than 75%
C. At least 75%, but less than 80%
D. At least 80%, but less than 85%
E. At least 85%

3.40 (3 points) Emails arrive at a Poisson rate of one per 2 minutes.

Nancy determines the exact probability that 4 or more emails will arrive in the next 6 minutes.
Sluggo estimates the same probability by applying the Normal Approximation to the distribution of the fourth event time. What is the difference in their estimates?

A. Less than 2%
B. At least 2%, but less than 3%
C. At least 3%, but less than 4%
D. At least 4%, but less than 5%
E. At least 5%

3.41 (1 point) For a homogeneous Poisson Process, the probability that more than 5 days elapses between events is 0.30.
Calculate the expected number of events in the next 100 days.

A. Less than 10
B. At least 10, but less than 15
C. At least 15, but less than 20
D. At least 20, but less than 25
E. At least 25

3.42 (2 points) Very large asteroids hit the moon at a Poisson rate of one every 10 million years.
What is the probability that more than 110 very large asteroids will hit the moon in the next 1000 million years? Use the Normal Approximation.

A. 15% B. 17% C. 19% D. 21% E. 23%

3.43 (2 points) Major earthquakes (of magnitude 8 and higher) occur according to a homogeneous Poisson process, with a rate of 1.36 per year.
Calculate the variance of the waiting time until the tenth major earthquake occurs.

A. Less than 4.8
B. At least 4.8, but less than 5.0
C. At least 5.0, but less than 5.2
D. At least 5.2, but less than 5.4
E. At least 5.4

3.44 (2, 5/90, Q.43) (1.7 points) Customers arrive randomly and independently at a service window, and the time between arrivals has an exponential distribution with a mean of 12 minutes. Let X equal the number of arrivals per hour. What is P[X = 10]?

A. $10e^{-12}/10!$ B. $10^{12}e^{-10}/10!$ C. $12^{10}e^{-10}/10!$ D. $12^{10}e^{-12}/10!$ E. $5^{10}e^{-5}/10!$

3.45 (Course 151 Sample Exam #2, Q.7) (0.8 points)
For a claim number process $\{N(t), t \geq 0\}$ you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function

$F(t) = 1 - e^{-3t}, t \geq 0.$

Determine the probability of exactly 4 claims in an interval of length 2.

(A) 0.11 (B) 0.13 (C) 0.15 (D) 0.17 (E) 0.19

3.46 (1, 11/00, Q.34) (1.9 points) The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.
What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

(A) 0.15 (B) 0.34 (C) 0.43 (D) 0.57 (E) 0.66

3.47 (IOA 101, 4/01, Q.3) (2.25 points) Suppose that the occurrence of events which give rise to claims in a portfolio of automobile insurance policies can be modeled as follows: the events occur through time at random, at rate μ per hour. Then the number of events which occur in a given period of time has a Poisson distribution (you are given this).
Show that the time between two consecutive events occurring has an exponential distribution with mean $1/\mu$ hours.

3.48 (1, 5/01, Q.22) (1.9 points) The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively.
What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

(A) $(1 - e^{-2/3} - e^{-1/2} + e^{-7/6})/18$

(B) $e^{-7/6}/18$

(C) $1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$

(D) $1 - e^{-2/3} - e^{-1/2} + e^{-1/3}$

(E) $1 - e^{-2/3}/3 - e^{-1/2}/6 + e^{-7/6}/18$

3.49 (IOA 101, 9/01, Q.6) (3.75 points)
(i) (1.5 points) The occurrence of claims in a group of 200 policies is modeled such that the probability of a claim arising in the next year is 0.015 independently for each policy.
Each policy can give rise to a maximum of one claim.
Calculate an approximate value for the probability that more than 10 claims arise from this group of policies in the next year by approximating via a Poisson.
Leave your answer in terms of an Incomplete Gamma Function.
(ii) (2.25 points) The occurrence of claims in a group of 2000 policies is modeled such that the probability of a claim arising in the next year is 0.015 independently for each policy.
Each policy can give rise to a maximum of one claim.
Using the Normal Approximation, calculate an approximate value for the probability that more than 40 claims arise from this group of policies in the next year.

3.50 (CAS3, 11/05, Q.28) (2.5 points) Big National Bank has 3 teller windows open for customer service. Each teller services customers at a Poisson rate of 6 customers per hour.
There is a single line to wait for the next available teller and all tellers are currently serving customers.
If there are 2 people in line when the next customer arrives, calculate the probability that he must wait more than 10 minutes for the next available teller.
A. Less than 30%
B. At least 30%, but less than 40%
C. At least 40%, but less than 50%
D. At least 50%, but less than 60%
E. At least 60%

3.51 (CAS3, 11/06, Q.27) (2.5 points)
A customer service operator accepts calls continuously throughout the work day.
The length of each call is exponentially distributed with an average of 3 minutes.
Calculate the probability that at least one call will be completed in the next 2 minutes.
A. Less than 0.50
B. At least 0.50, but less than 0.55
C. At least 0.55, but less than 0.60
D. At least 0.60, but less than 0.65
E. At least 0.65

3.52 (SOA M, 11/06, Q.8) (2.5 points)
The time elapsed between claims processed is modeled such that V_k represents the time elapsed between processing the k-1th and kth claim.
(V_1 = time until the first claim is processed).
You are given:
(i) V_1, V_2,... are mutually independent.
(ii) The pdf of each V_k is $f(t) = 0.2\ e^{-0.2t}$, $t > 0$, where t is measured in minutes.
Calculate the probability of at least two claims being processed in a ten minute period.
(A) 0.2 (B) 0.3 (C) 0.4 (D) 0.5 (E) 0.6

3.53 (CAS3, 5/07, Q.1) (2.5 points) You are given the following information:

- The number of wild fires per day in a state follows a Poisson distribution.
- The expected number of wild fires in a thirty-day time period is 15.

Calculate the probability that the time between the eighth and ninth fire will be greater than three days.
A. Less than 20%
B. At least 20%, but less than 25%
C. At least 25%, but less than 30%
D. At least 30%, but less than 35%
E. At least 35%

3.54 (SOA MLC, 5/07, Q.5) (2.5 points) Heart/Lung transplant claims in 2007 have interevent times that are independent with a common distribution which is exponential with mean one month.
As of the end of January, 2007 no transplant claims have arrived.
Calculate the probability that at least three Heart/Lung transplant claims will have arrived by the end of March, 2007.
(A) 0.18 (B) 0.25 (C) 0.32 (D) 0.39 (E) 0.45

3.55 (SOA MLC, 5/07, Q.26) (2.5 points)
A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 4 per day, and that a specimen fails at the time of the 289th such mistake.
This theory explains the only cause of failure.
T is the time-of-failure random variable in days for a newborn specimen.
Using the normal approximation, calculate the probability that $T > 68$.
(A) 0.84 (B) 0.86 (C) 0.88 (D) 0.90 (E) 0.92

3.56 (CAS3, 11/07, Q.1) (2.5 points)
You are given the following information about the interarrival times for tornadoes in county XYZ.

- The waiting time in days between tornadoes follows an exponential distribution and remains constant throughout the year.
- The probability that more than 30 days elapses between tornadoes is .60.

Calculate the expected number of tornadoes in the next 90 days.
A. Less than 1.0
B. At least 1.0, but less than 1.5
C. At least 1.5, but less than 2.0
D. At least 2.0, but less than 2.5
E. At least 2.5

3.57 (CAS3, 11/07, Q.2) (2.5 points)
Car crashes occur according to a Poisson process at a rate of 2 per hour.
There have been 12 crashes between 9:00 AM and 10:00 AM.
Given that it is now 10:00 AM, at what time do you expect the 13th crash?
A. Before 10:05 AM
B. On or after 10:05 AM, but before 10:15 AM
C. On or after 10:15 AM, but before 10:25 AM
D. On or after 10:25 AM, but before 10:35 AM
E. On or after 10:35 AM

3.58 (CAS3L, 11/10, Q.10) (2.5 points) Hurricanes make landfall according to a homogeneous Poisson process, with a rate of 1.5 per month.
Calculate the variance (in months squared) of the waiting time until the third hurricane makes landfall.
A. Less than 1.35
B. At least 1.35, but less than 1.45
C. At least 1.45, but less than 1.55
D. At least 1.55, but less than 1.65
E. At least 1.65